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TECHNICAL MEMORANDUM

CHARACTERIZATION AND MATHEMATICAL REPRESENTATION OF LINE SCANNING IMAGE SYSTEMS

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ABSTRACT

By the use of basic techniques which have been developed for treating linear filters in network theory, models and the mathematical relationships of scanning are derived. To overcome the limitations of the one-dimensional and two-dimensional Fourier integral analyses of the line scanning system, a semi-discrete method of analysis is developed. This method is used to evaluate more precisely the system performance when the output signal is inherently discrete and when post filtering is performed.

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TECHNICAL MEMORANDUM

I. INTRODUCTION

The application of techniques of network theory and information theory to the study of optical systems has received a considerable amount of attention.[1-8] In this memorandum we utilize these basic techniques as a tool for the analysis of a line scanning image system, and to develop models and derive relationships which are necessary for the evaluation of system performance.

The analysis presented here avoids the limitations of the one-dimensional and two-dimensional Fourier integral analyses of the line scanning system (see Appendix). We interpret the line scanning process as a one-dimensional sampling operation with period Δy of a continuous two-dimensional signal, as shown in Figure 1.

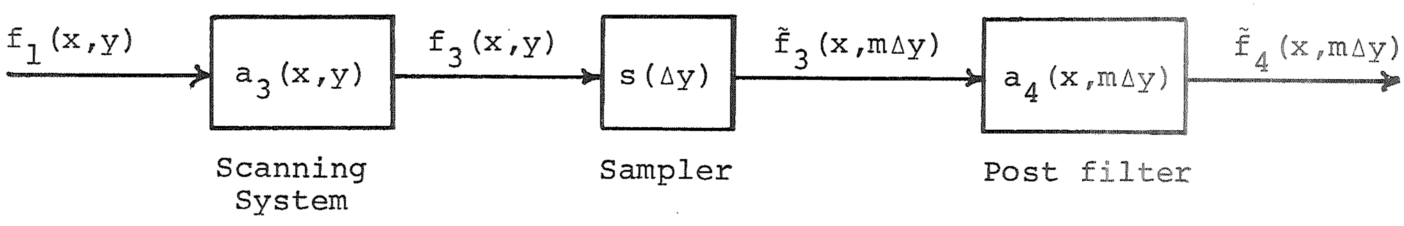

FIGURE 1 - SCANNING AS A SAMPLING PROCESS

In Figure 1, $f_1(x,y)$ is the input signal and $a_3(x,y)$ is the point source response of the scanning system of Figure A given in the Appendix. The sampler is viewed as an operation characteristic of the scanning process which limits the output $f_3(x,y)$ to record the input signal on a set of parallel lines, continuous in x over some finite length and sampled periodically at discrete positions in y with period Δy. It should be noted that this interpretation is more realistic than that of the continuous two-dimensional output since the size of the scanning aperture and the distance between scan lines of the y coordinate can never be zero. The purpose of the post filter $a_4(x,m\Delta y)$ is to process the signal $\tilde{f}_3(x,m\Delta y)$ to recover the original input signal which is contaminated by two-dimensional noise introduced by the system and scanning process.

As stated in the last paragraph, a more realistic interpretation of a scanning process is to consider the input and output signals as semi-discrete (i.e., a set of parallel lines). Noting that the input-output relationships of semi-discrete signals cannot be expressed in terms of two-dimensional continuous convolution, we shall derive these relationships in terms of semi-discrete representation in the following sections.

II. CHARACTERIZATION OF SIGNAL BY CORRELATION FUNCTIONS

Let $A_f(x_1,x_2;y_1,y_2)$ denote the autocorrelation function of a continuous spatial signal f(x,y), and define it as

$$A_f(x_1,x_2;y_1,y_2) = E[f(x_1,y_1)f(x_2,y_2)] \tag{1}$$

where E denotes the ensemble expectation, and (x_1,y_1) a point in the (x,y) plane. For a spatially stationary signal, (1) reduces to

$$A_f(x,y) = E[f(x+a,y+b)f(a,b)] = E[f_+f] \tag{2}$$

where the + subscript on signal variable denotes the shift required by correlation and f denotes f(x,y).

The ensemble autocorrelation function for a semi-discrete stationary signal is defined as

$$\tilde{A}_f(x,m\Delta y) = E[\tilde{f}(x+a,m\Delta y+n\Delta y)\tilde{f}(a,n\Delta y)] = E[\tilde{f}_+\tilde{f}] \qquad (3)$$

where "~" denotes semi-discrete function, continuous in x, discrete in y.

Cross-correlation functions for continuous stationary signals are defined as

$$C_{f_{1+}f_2}(x,y) = E[f_1(x+a,y+b)f_2(a,b)] = E[f_{1+}f_2] \qquad (4)$$

and for semi-discrete stationary signals are

$$\tilde{C}_{f_{1+}f_2}(x,m\Delta y) = E[\tilde{f}_1(x+a,m\Delta y+n\Delta y)\tilde{f}_2(a,n\Delta y)] = E[\tilde{f}_{1+}\tilde{f}_2]. \qquad (5)$$

The frequency spectra for the correlation functions (2) to (5) are defined as:

$$S_f(w_x,w_y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A_f(x,y)\exp[-2\pi j(w_x x+w_y y)]dxdy \qquad (6)$$

$$\tilde{S}_f(w_x,w_y) = \int_{-\infty}^{\infty}\sum_{m=-\infty}^{\infty} \tilde{A}_f(x,m\Delta y)\exp[-2\pi j(w_x x+w_y m\Delta y)]dx\Delta y \qquad (7)*$$

---

*Equations (7) and (9) can be readily obtained by defining the sampling operation as $S[G_f(x,y)] = \sum_{m=-\infty}^{\infty} \delta(\frac{y}{\Delta y} - m)G_f(x,y)$ and assuming a sampling process such that $G_f(x,m\Delta y) = \tilde{G}_f(x,m\Delta y)$. This is done so that the forms of the continuous and semi-discrete relations (6) to (9) are preserved.

$$S_{f_1+f_2}(w_x,w_y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} C_{f_1+f_2}(x,y)\exp[-2\pi j(w_x x+w_y y)]\,dxdy \qquad (8)$$

$$\tilde{S}_{f_1+f_2}(w_x,w_y) = \int_{-\infty}^{\infty}\sum_{m=-\infty}^{\infty} \tilde{C}_{f_1+f_2}(x,m\Delta y)\exp[-2\pi j(w_x x+w_y m\Delta y)]\,dx\Delta y \qquad (9)$$

respectively. Using these definitions, the inverse relationships of (6) and (8) are derived in Reference [1] as

$$A_f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S_f(w_x,w_y)\exp[2\pi j(w_x x+w_y y)]\,dw_x dw_y \qquad (10)$$

and

$$C_{f_1+f_2}(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S_{f_1+f_2}(w_x,w_y)\exp[2\pi j(w_x x+w_y y)]\,dw_x dw_y \qquad (11)$$

respectively.

Now we shall proceed to derive the inverse relationship for the semi-discrete two-dimensional frequency spectra of (7) and (9). Consider first the frequency spectrum of the semi-discrete two-dimensional autocorrelation function given in (7).

Multiplying both sides of (7) by $\exp[2\pi j w_y p\Delta y]$ and assuming that the order of integration and summation can be reversed (this assumption implies term-by-term integration in (7) is allowed) we have

$$\tilde{S}_f(w_x,w_y)\exp[2\pi j w_y p\Delta y] = \sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{A}_f(x,m\Delta y)\exp[-2\pi j w_x x]\,dx$$
$$\exp[2\pi j(p-m)w_y\Delta y]\,\Delta y \,. \qquad (12)$$

Integrate both sides of (12) over a period of $\frac{1}{\Delta y}$ with respect to w_y

$$\int_{-\frac{1}{2\Delta y}}^{\frac{1}{2\Delta y}} \tilde{S}_f(w_x,w_y)\exp[2\pi j w_y p\Delta y]\,dw_y = \Delta y \int_{-\frac{1}{2\Delta y}}^{\frac{1}{2\Delta y}} \left[\sum_{m=-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \tilde{A}_f(x,m\Delta y) \exp(-2\pi j w_x x)\,dx \right] \exp(2\pi j (p-m) w_y \Delta y) \right] dw_y \,. \quad (13)$$

Again assuming that the order of integration and summation can be changed, the right-hand side of (13) can be rewritten as

$$\Delta Y \sum_{m=-\infty}^{\infty} \int_{-\frac{1}{2\Delta y}}^{\frac{1}{2\Delta y}} \exp[2\pi j (p-m) w_y \Delta y]\,dw_y \left[\int_{-\infty}^{\infty} \tilde{A}_f(x,m\Delta y)\exp[-2\pi j w_x x]\,dx \right]. \quad (14)$$

Substituting $z = 2\pi\Delta y w_y$ and $dz = 2\pi\Delta y dw_y$ into (14), it becomes

$$\frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \int_{-\pi}^{\pi} \exp[j(p-m)z]\,dz \left[\int_{-\infty}^{\infty} \tilde{A}_f(x,m\Delta y)\exp[-2\pi j w_x x]\,dx \right] . \quad (15)$$

Noting that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \exp[j(p-m)z]\,dz = \begin{cases} 1 & p = m \\ 0 & p \neq m \end{cases} \quad (16)$$

(15) reduces to

$$\int_{-\infty}^{\infty} \tilde{A}_f(x,p\Delta y)\exp[-2\pi j w_x x]\,dx \,. \quad (17)$$

Thus, (13) can now be written as

$$\begin{aligned} \Gamma(w_x) &= \int_{-\frac{1}{2\Delta y}}^{\frac{1}{2\Delta y}} \tilde{S}_f(w_x,w_y)\exp[2\pi j w_y p\Delta y]\,dw_y \\ &= \int_{-\infty}^{\infty} \tilde{A}_f(x,p\Delta y)\exp[-2\pi j w_x x]\,dx \end{aligned} \quad (18)$$

where $\Gamma(w_x)$ is the x Fourier transform of $\tilde{A}_f(x,p\Delta y)$. Then applying the well-known one-dimensional inverse Fourier transform relation and replacing p by m, we have

$$\tilde{A}(x,m\Delta y) = \int_{-\infty}^{\infty} \Gamma(w_x)\exp[2\pi j w_x x]\,dw_x \qquad (19)$$

or

$$\tilde{A}_f(x,m\Delta y) = \int_{-\infty}^{\infty}\int_{-\frac{1}{2\Delta y}}^{\frac{1}{2\Delta y}} \tilde{S}_f(w_x,w_y)\exp[2\pi j(w_x x+w_y m\Delta y)]\,dw_y dw_x \;. \qquad (20)$$

Using the same procedure and the definition of (9) the semi-discrete two-dimensional cross correlation function can be written

$$\tilde{C}_{f_1+f_2}(x,m\Delta y) = \int_{-\infty}^{\infty}\int_{-\frac{1}{2\Delta y}}^{\frac{1}{2\Delta y}} \tilde{S}_{f_1+f_2}(w_x,w_y)\exp[2\pi j(w_x x+w_y m\Delta y)]\,dw_y dw_x \;. \qquad (21)$$

Utilizing the well-known mean square value theorem in communication theory, the mean square value for a semi-discrete signal can be obtained directly from (3) and (20)

$$E[\tilde{f}^2] = \tilde{A}_f(0,0) = \int_{-\infty}^{\infty}\int_{-\frac{1}{2\Delta y}}^{\frac{1}{2\Delta y}} \tilde{S}_f(w_x,w_y)\,dw_y dw_x \;. \qquad (22)$$

III. <u>RELATIONSHIP BETWEEN THE CONTINUOUS AND SEMI-DISCRETE SPECTRA</u>

Assume a sampling process that exists in the following relationship,

$$\tilde{A}_f(x,m\Delta y) = A_f(x,m\Delta y) \;. \qquad (23)$$

Using the definition of (10) and equation (23), we obtain

$$\tilde{A}_f(x,m\Delta y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S_f(w_x,w_y)\exp[2\pi j(w_x x+w_y m\Delta y)]\,dw_x dw_y \ . \qquad (24)$$

First assume each integration of (24) converges uniformly with respect to the infinite limits; then rewrite the w_y integration of the right-hand side of (24) as the sum of sub-integrals of period $\frac{1}{\Delta y}$

$$\tilde{A}_f(x,m\Delta y) = \int_{-\infty}^{\infty}\left[\sum_{k=-\infty}^{\infty}\int_{\frac{k}{\Delta y}-\frac{1}{2\Delta y}}^{\frac{k}{\Delta y}+\frac{1}{2\Delta y}} S_f(w_x,w_y)\exp[2\pi j(w_x x+w_y m\Delta y)]\,dw_y\right]dw_x \ . \qquad (25)$$

Substituting $w_y = \frac{k}{\Delta y} + w_y'$ and noting that $e^{j2\pi mk} = 1$ if m and k are integers, (25) becomes

$$\tilde{A}_f(x,m\Delta y) = \int_{-\infty}^{\infty}\left[\sum_{k=-\infty}^{\infty}\int_{-\frac{1}{2\Delta y}}^{\frac{1}{2\Delta y}} S_f(w_x,w_y'+\frac{k}{\Delta y})\exp[2\pi j(w_x x+w_y' m\Delta y)]\,dw_y'\right]dw_x \ . \qquad (26)$$

Replacing w_y for w_y' and reversing the order of integration and summation as before, we obtain

$$\tilde{A}_f(x,m\Delta y) = \int_{-\infty}^{\infty}\int_{-\frac{1}{2\Delta y}}^{\frac{1}{2\Delta y}} \sum_{k=-\infty}^{\infty} S_f(w_x,w_y+\frac{k}{\Delta y})\exp[2\pi j(w_x x+w_y m\Delta y)]\,dw_y dw_x \ . \qquad (27)$$

Comparing (27) with (20) we derive the relationship between the continuous and semi-discrete spectra as

$$\tilde{S}_f(w_x,w_y) = \sum_{k=-\infty}^{\infty} S_f(w_x,w_y+\frac{k}{\Delta y}) \ . \qquad (28)$$

Similarly, the relationship between the continuous and semi-discrete spectra for the cross correlation function is

$$\tilde{S}_{f_1+f_2}(w_x,w_y) = \sum_{k=-\infty}^{\infty} S_{f_1+f_2}(w_x,w_y+\frac{k}{\Delta y}) \qquad (29)$$

respectively.

IV. INPUT-OUTPUT RELATIONSHIPS OF A LINEAR FILTER

If an input signal $f_1(x,y)$ passes through a linear filter with system response function $h(x,y)$, then the output signal $f_2(x,y)$ can be expressed as a continuous convolution

$$f_2(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_1(x',y')h(x-x',y-y')dx'dy' = h * f_1 \qquad (30)$$

where $*$ denotes convolution. Similarly, if a semi-discrete signal $\tilde{f}_1(x,m\Delta y)$ passes through a linear semi-discrete filter with system response function $\tilde{h}(x,m\Delta y)$ then the semi-discrete output signal $\tilde{f}_2(x,m\Delta y)$ may be determined by a semi-discrete convolution,

$$\tilde{f}_2(x,m\Delta y) = \int_{-\infty}^{\infty}\sum_{i=-\infty}^{\infty} \tilde{f}_1(x',i\Delta y)\tilde{h}(x-x',m\Delta y-i\Delta y)dx'\Delta y = \tilde{h} \tilde{*} \tilde{f}_1 \qquad (31)$$

where $\tilde{*}$ denotes semi-discrete convolution.

Using the input-output relationships of (30) and (31) and assuming the input signal is stationary, we proceed to derive the input-output relationships corresponding to the two-dimensional continuous and semi-discrete correlation functions and their Fourier spectra when passed through a linear space-invariant (or, equivalently, isoplanatic) filter.

From (2) and (30), the output autocorrelation function is

$$A_{f_2}(x,y) = E[f_2(x+a,y+b)f_2(a,b)] \quad (32)$$

$$= E\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_1(x',y')h(x+a-x',y+b-y')dx'dy' \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_1(u,v)h(a-u,b-v)dudv\right]$$

or, rewriting,

$$A_{f_2}(x,y) = E\left[\int\int_{-\infty}^{\infty}\int\int f_1(x',y')f_1(u,v)h(x+a-x',y+b-y')h(a-u,b-v) dx'dy'dudv\right] . \quad (33)$$

Bring the expectation inside the integral and let

$$x' - u = s$$
$$y' - v = t$$
$$x' = p$$
$$y' = q$$

we obtain

$$A_{f_2}(x,y) = \int\int_{-\infty}^{\infty}\int\int h(x+a-p,y+b-q)h(a-p+s,b-q+t) E[f_1(u+s,v+t)f_1(u,v)]dsdtdpdq . \quad (34)$$

From (2), (34) becomes

$$A_{f_2}(x,y) = \int \int_{-\infty}^{\infty} \int \int h(w,z)h(w-x+s,z-y+t)A_{f_1}(s,t)dsdtdwdz \ . \quad (36)$$

Noting that

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(w,z)h(w-u,z-v)dwdz = h * h_- = |h|(u,v) \quad (37)$$

where $h_- = h(-x,-y)$. Thus (36) becomes

$$A_{f_2}(x,y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |h|(x-s,y-t)A_{f_1}(s,t)dsdt = h * h_- * A_{f_1} \ . \quad (38)$$

The derivation of the input-output relationship in terms of a two-dimensional semi-discrete autocorrelation function is similar to the case of the two-dimensional continuous autocorrelation function. From (3) and (31), the output semi-discrete autocorrelation function when an input semi-discrete function is transmitted through a linear space-invariant filter is

$$\tilde{A}_{f_2}(x,m\Delta y) = E\left[\int_{-\infty}^{\infty} \sum_{i=-\infty}^{\infty} \tilde{f}_1(x',i\Delta y)\tilde{h}(x+a-x',m\Delta y+n\Delta y-i\Delta y)dx'\Delta y \int_{-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \tilde{f}_1(u,j\Delta y)\tilde{h}(a-u,n\Delta y-j\Delta y)du\Delta y \right] \ . \quad (39)$$

Noting that the integrations and summations are all over different variables, (39) can be rewritten as

$$\tilde{A}_{f_2}(x,m\Delta y) = E\left[\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \tilde{f}_1(x',i\Delta y)\tilde{f}_1(u,j\Delta y) \tilde{h}(x+a-x',m\Delta y+n\Delta y-i\Delta y)\tilde{h}(a-u,n\Delta y-j\Delta y)dx'du(\Delta y)^2 \right] \ . \quad (40)$$

Bringing the expectation inside the integral and letting

$$x' - u = w \qquad x' = \underline{v}$$

$$i\Delta y - j\Delta y = p\Delta y \qquad i\Delta y = z\Delta y$$

we obtain

$$\tilde{A}_{f_2}(x,m\Delta y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sum_{z=-\infty}^{\infty}\sum_{p=-\infty}^{\infty} \tilde{h}(x+a-\underline{v},m\Delta y+n\Delta y-z\Delta y)$$

$$\tilde{h}(a-\underline{v}+w,n\Delta y-z\Delta y+p\Delta y)E[\tilde{f}_1(u+w,j\Delta y+p\Delta y)\tilde{f}_1(u,j\Delta y)]$$

$$d\underline{v}dw(\Delta y)^2 . \tag{41}$$

From (3), (41) becomes

$$\tilde{A}_{f_2}(x,m\Delta y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sum_{z=-\infty}^{\infty}\sum_{p=-\infty}^{\infty} \tilde{h}(x+a-\underline{v},m\Delta y+n\Delta y-z\Delta y)$$

$$\tilde{h}(a-\underline{v}+w,n\Delta y-z\Delta y+p\Delta y)\tilde{A}_{f_1}(w,p\Delta y)d\underline{v}dw(\Delta y)^2 . \tag{42}$$

Considering only the integral

$$\int_{-\infty}^{\infty}\sum_{z=-\infty}^{\infty} \tilde{h}(x+a-\underline{v},m\Delta y+n\Delta y-z\Delta y)\tilde{h}(a-\underline{v}+w,n\Delta y-z\Delta y+p\Delta y)d\underline{v}\Delta y \tag{43}$$

of (42) and letting

$$x + a - \underline{v} = k$$

$$m\Delta y + n\Delta y - z\Delta y = \ell\Delta y$$

the integral (43) reduces to

$$\int_{-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} \tilde{h}(k,\ell\Delta y)\tilde{h}(k-x+w,\ell\Delta y-m\Delta y+p\Delta y)dk\Delta y = |\tilde{h}|(x-w,m\Delta y-p\Delta y) \quad (44)$$

where

$$|\tilde{h}| = \tilde{h} \tilde{*} \tilde{h}_- \quad (45)$$

and

$$\tilde{h}_- = \tilde{h}_-(x,m\Delta y) = \tilde{h}(-x,-m\Delta y).$$

Hence, (42) becomes

$$\tilde{A}_{f_2}(x,m\Delta y) = \int_{-\infty}^{\infty} \sum_{p=-\infty}^{\infty} |\tilde{h}|(x-w,m\Delta y-p\Delta y)\tilde{A}_{f_1}(w,p\Delta y)dw\Delta y \quad (46)$$

$$= |\tilde{h}| \tilde{*} \tilde{A}_{f_1} = \tilde{h} \tilde{*} \tilde{h}_- \tilde{*} \tilde{A}_{f_1} \quad (46)$$

The two-dimensional cross correlation function between the input and output signals can be obtained from (4) and (30)

$$C_{f_1+f_2}(x,y) = E[f_1(x+a,y+b) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_1(x',y')h(a-x',b-y')dx'dy'] . \quad (47)$$

First note that $f_1(x+a,y+b)$ is not a function of x' and y'. Bringing it and the expectation inside the integral (47) becomes

$$C_{f_1+f_2}(x,y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(a-x',b-y')E[f_1(x+a,y+b)f_1(x',y')]dx'dy'. \quad (48)$$

Substituting

$$u = a-x'$$

$$\underline{v} = b-y'$$

into (48), we have

$$C_{f_{1+}f_2}(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(u,\underline{v})E[f_1(x+u+x',y+\underline{v}+y')f_1(x',y')]dud\underline{v} \qquad (49)$$

or

$$C_{f_{1+}f_2}(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(u,\underline{v})A_{f_1}(u+x,\underline{v}+y)dud\underline{v}$$

$$= A_{f_1}(x,y) * h_-(x,y) \ . \qquad (50)$$

Similarly, the reverse continuous two-dimensional cross correlation, the semi-discrete two-dimensional cross correlation and the semi-discrete reverse two-dimensional cross correlation functions can be shown to be

$$C_{f_1f_{2+}}(x,y) = h(x,y) * A_{f_1}(x,y) \qquad (51)$$

$$\tilde{C}_{f_{1+}f_2}(x,m\Delta y) = \tilde{h}_-(x,m\Delta y) \tilde{*} \tilde{A}_{f_1}(x,m\Delta y)$$

$$= \int_{-\infty}^{\infty}\sum_{p=-\infty}^{\infty} \tilde{h}(w,p\Delta y)\tilde{A}_{f_1}(x+w,m\Delta y+p\Delta y)dw\Delta y \qquad (52)$$

and

$$\tilde{C}_{f_1f_{2+}}(x,m\Delta y) = \tilde{h}(x,m\Delta y) \tilde{*} \tilde{A}_{f_1}(x,m\Delta y)$$

$$= \int_{-\infty}^{\infty}\sum_{p=-\infty}^{\infty} \tilde{h}(w,p\Delta y)\tilde{A}_{f_1}(x-w,m\Delta y-p\Delta y)dw\Delta y \qquad (53)$$

respectively.

Thus, the input-output relationships in terms of the correlation functions and system response function can be summarized as below:

$$
\begin{aligned}
A_{f_2} &= A_{f_1} * h * h_- \\
\tilde{A}_{f_2} &= \tilde{A}_{f_1} \tilde{*} \tilde{h} \tilde{*} \tilde{h}_- \\
C_{f_{1+}f_2} &= A_{f_1} * h_- \\
C_{f_1 f_{2+}} &= A_{f_1} * h \\
\tilde{C}_{f_{1+}f_2} &= \tilde{A}_{f_1} \tilde{*} \tilde{h}_- \\
\tilde{C}_{f_1 f_{2+}} &= \tilde{A}_{f_1} \tilde{*} \tilde{h}
\end{aligned}
\qquad (54)
$$

where the + subscript on signal variable represents the shift required by correlation, the "~" denotes semi-discrete operation, h is the system response function, $h_-(x,y) = h(-x,-y)$ and $\tilde{h}_-(x,m\Delta y) = h(-x,-m\Delta y)$.

Using the two-dimensional Fourier transform techniques, (54) can be expressed in the spatial frequency domain as

$$
\begin{aligned}
S_{f_2} &= S_{f_1} |H|^2 \\
\tilde{S}_{f_2} &= \tilde{S}_{f_1} |\tilde{H}|^2 \\
S_{f_{1+}f_2} &= S_{f_1} \overline{H} \\
S_{f_1 f_{2+}} &= S_{f_1} H \\
\tilde{S}_{f_{1+}f_2} &= \tilde{S}_{f_1} \overline{\tilde{H}} \\
\tilde{S}_{f_1 f_{2+}} &= \tilde{S}_{f_1} \tilde{H}
\end{aligned}
\qquad (55)
$$

where H is the two-dimensional Fourier transform of h, $\overline{H}$ is the conjugate of H, and $|H|^2$ equals $H\overline{H}$.

V. AUTOCORRELATION FUNCTION OF THE ERROR SIGNAL WITH POST FILTER INCLUDED

The model given in Figure 2 enables one to evaluate system performance and to employ post filtering techniques. In terms of semi-discrete signals the error is

$$\tilde{\varepsilon} = \tilde{f}_1 - \tilde{f}_4 = \tilde{f}_1 - (\tilde{f}_3 + \tilde{n}) * \tilde{a}_4 \qquad (56)$$

where the variables are defined in Figure 2.

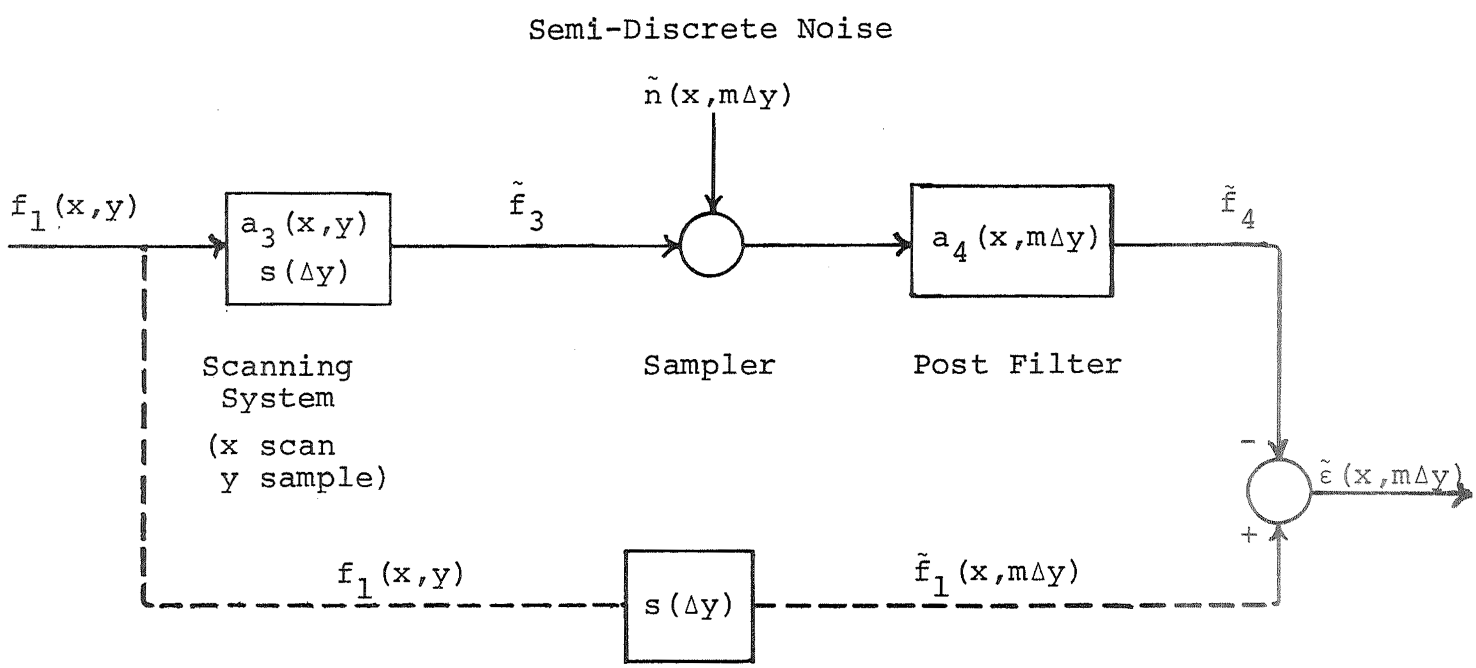

FIGURE 2 - MODEL FOR PERFORMANCE EVALUATION AND POST FILTERING

From (3) and (56), the autocorrelation function of the error signal is

$$\begin{aligned} \tilde{A}_\varepsilon &= E[\tilde{\varepsilon}_+ \tilde{\varepsilon}] \\ &= E\{[\tilde{f}_{1+} - (\tilde{f}_3 \tilde{*} \tilde{a}_4)_+ - (\tilde{n} \tilde{*} \tilde{a}_4)_+][\tilde{f}_1 - (\tilde{f}_3 \tilde{*} \tilde{a}_4) - (\tilde{n} \tilde{*} \tilde{a}_4)]\} \; . \end{aligned} \tag{57}$$

Expanding (57) and assuming the signal and noise are uncorrelated and have zero mean, that is

$$E[\tilde{n}_+ \tilde{f}_1] = E[\tilde{n}\tilde{f}_{1+}] = E[\tilde{n}] = E[\tilde{f}] = 0 \tag{58}$$

we obtain

$$\begin{aligned} \tilde{A}_\varepsilon = \; & E[(\tilde{f}_3 \tilde{*} \tilde{a}_4)_+ (\tilde{f}_3 \tilde{*} \tilde{a}_4)] + E[(\tilde{n} \tilde{*} \tilde{a}_4)_+ (\tilde{n} \tilde{*} \tilde{a}_4)] - E[\tilde{f}_{1+} (\tilde{f}_3 \tilde{*} \tilde{a}_4)] \\ & - E[\tilde{f}_1 (\tilde{f}_3 \tilde{*} \tilde{a}_4)_+] + E[\tilde{f}_{1+} \tilde{f}_1] \; . \end{aligned} \tag{59}$$

Applying (3) and (54) to (59), it reduces to

$$\tilde{A}_\varepsilon = (\tilde{A}_{f_3} + \tilde{A}_n) \tilde{*} \tilde{a}_4 \tilde{*} \tilde{a}_{4-} - \tilde{C}_{f_{1+} f_3} \tilde{*} \tilde{a}_{4-} - \tilde{C}_{f_1 f_{3+}} \tilde{*} \tilde{a}_4 + \tilde{A}_{f_1} \tag{60}$$

or in the spatial frequency domain, we apply (55) to (60) and obtain

$$\tilde{S}_\varepsilon = (\tilde{S}_{f_3} + \tilde{S}_n) |\tilde{A}_4|^2 - \tilde{S}_{f_{1+} f_3} \bar{\tilde{A}}_4 - \tilde{S}_{f_1 f_{3+}} \tilde{A}_4 + \tilde{S}_{f_1} \; . \tag{61}$$

Thus, we can evaluate the system performance by determining $\tilde{A}_\varepsilon$ using either (60) or (61). It should be noted that the system performance before post filter can be obtained by either (60), letting $\tilde{a}_4(x,m\Delta y) = \tilde{\delta}(x,m\Delta y)$, or (61), letting $\tilde{A}_4(x,m\Delta y) = 1$.

Up to this point, semi-discrete functions continuous in the x coordinate have been considered. Sampling in the y variable is a consequence of the line scanning process which gives functions discrete in y.

For purposes of data recording or transmission, the semi-discrete functions are often sampled along the x, or equivalently, the time axis. In the process, quantization noise is introduced. Subsequently, the semi-discrete functions are reconstructed in time by passing the sampled data through a low pass filter.

The model for the line scanning system mentioned above is shown in Figure 3. The correlation spectrum of the output signal of the reconstruction filter $a_4(x,y)$ can be expressed as

$$\tilde{S}_{f_5} = (\tilde{S}_{f_1} + \tilde{S}_{n_1})|\tilde{A}_1\tilde{A}_2\tilde{A}_3\tilde{A}_4|^2 + \tilde{S}_{n_2}|\tilde{A}_2\tilde{A}_3\tilde{A}_4|^2 + \tilde{S}_{n_3}|\tilde{A}_3\tilde{A}_4|^2 + \tilde{S}_{n_4}|\tilde{A}_4|^2 \tag{62}$$

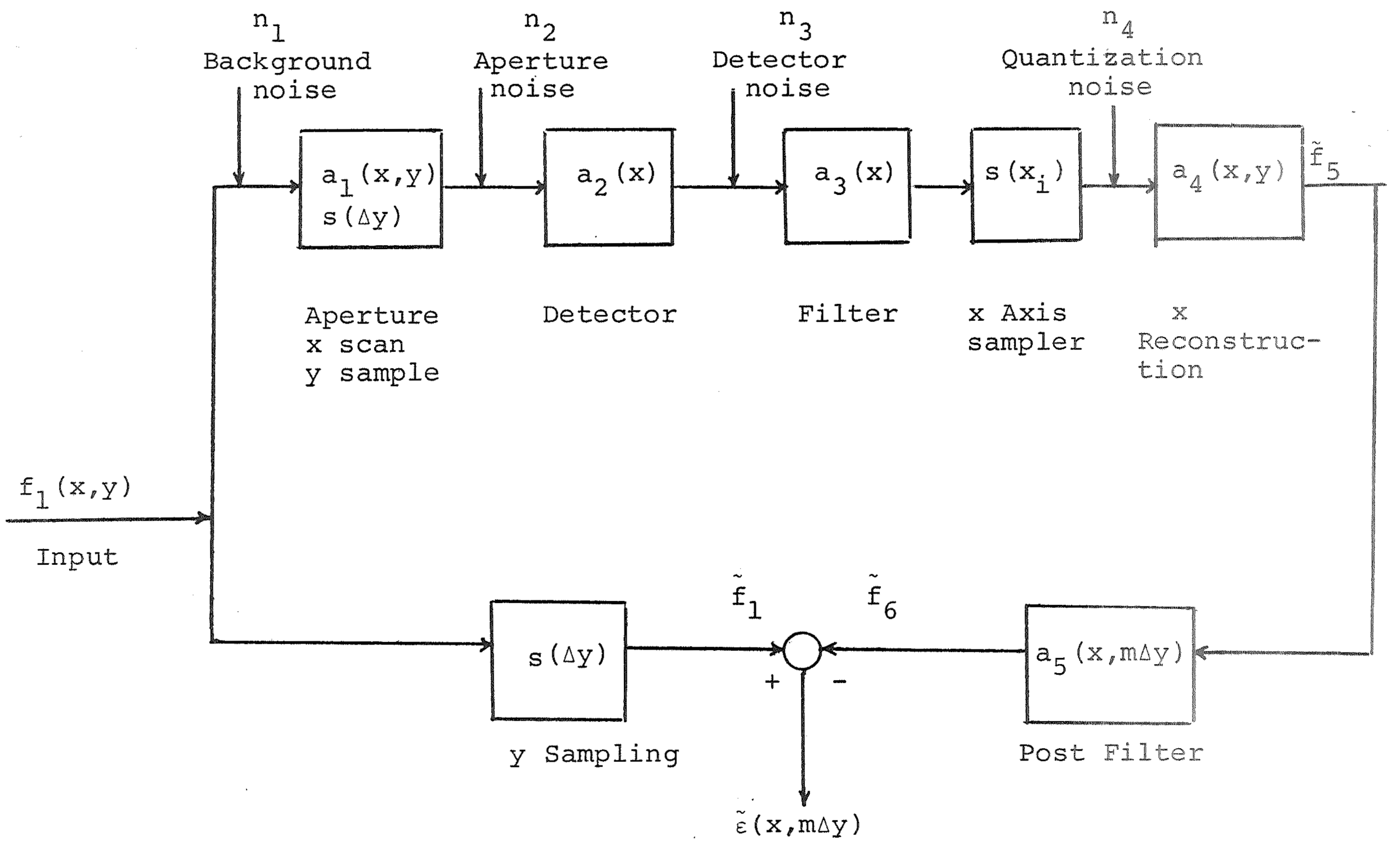

FIGURE 3 - MODEL OF A LINE SCANNING SYSTEM

Note that $\tilde{f}_5$ is associated with signal $\tilde{f}_3$ of Figure 2. The spectrum of the output of the post filter $\tilde{a}_5(x,m\Delta y)$ is

$$\tilde{S}_{f_6} = (\tilde{S}_{f_5})|\tilde{A}_5|^2 \tag{63}$$

and the error is

$$\tilde{\varepsilon} = \tilde{f}_1 - \tilde{f}_6 \ . \tag{64}$$

Thus from (61) we obtain

$$\tilde{S}_\varepsilon = (\tilde{S}_{f_5})|\tilde{A}_5|^2 - \tilde{S}_{f_{1+}f_5}\bar{\tilde{A}}_5 - \tilde{S}_{f_1 f_{5+}}\tilde{A}_5 + \tilde{S}_{f_1} \tag{65}$$

where $\tilde{S}_{f_5}$ is given in (62), $\tilde{S}_{f_{1+}f_5} = \tilde{S}_{f_1}\bar{\tilde{A}}_1\bar{\tilde{A}}_2\bar{\tilde{A}}_3\bar{\tilde{A}}_4$ and $\tilde{S}_{f_1 f_{5+}} = \tilde{S}_{f_1}\tilde{A}_1\tilde{A}_2\tilde{A}_3\tilde{A}_4$. Hence, by knowing the system functions or their spectra, the autocorrelation function of the input signal or its spectrum, and the autocorrelation functions of the various noises or their spectra, the system performance can be readily determined by (65).

VII. ACKNOWLEDGMENT

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Attachment

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APPENDIX

ONE-DIMENSIONAL FOURIER INTEGRAL ANALYSIS OF A LINE SCANNING PROCESS AND ITS LIMITATIONS

The line scanning image process shown in Figure A can be described mathematically as follows: a signal $f_1(x,y)$ is converted to

$$f_2(x,y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} a_1(x-x',y-y')f_1(x',y')dx'dy' \qquad \text{(A-1)}$$

at the output of the aperture, where $a_1(x,y)$ is the aperture response to a point source of light. Equation (A-1) can be interpreted as the relationship between the input signal at a point and the actual measured output due to the aperture spreading effect.

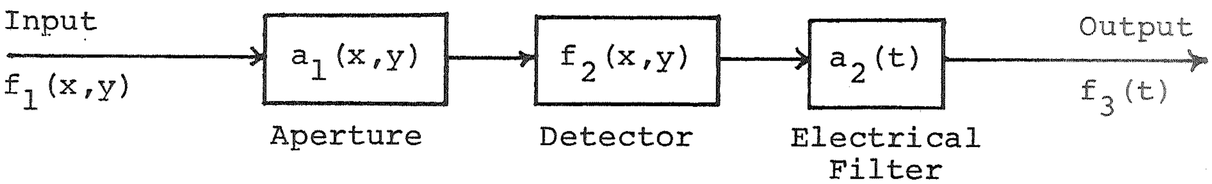

FIGURE A - A SCANNING PROCESS

Thus, if (x_1,y_1) denotes a point in the (x,y) plane, the output of the system at position (x_1,y_1) is

$$f_2(x_1,y_1) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} a_1(x_1-x',y_1-y')f_1(x',y')dx'dy' \qquad \text{(A-2)}$$

where x' and y' are in the aperture coordinate system. By assuming the aperture moves with respect to the surface of the input plane with a constant velocity (v) and the system output is a one-dimensional electrical signal, it is possible to analyze the scanning process in terms of its temporal response on a single line scan. To illustrate this method, first note that two-dimensional convolution in the spatial domain reduces to multiplication in the spatial frequency domain; thus equation (A-1) becomes

$$f_2(x,y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} A_1(w_x,w_y) F_1(w_x,w_y) \exp[2\pi j(w_x x + w_y y)] dw_x dw_y \qquad \text{(A-3)}$$

where $A_1(w_x,w_y)$ is the transfer function of the optical aperture and $F_1(w_x,w_y)$ is the two-dimensional Fourier transform of the input signal.

Now assume that the aperture scans the input signal in the x direction at a constant velocity (v) such that $x = vt$. Then the wave number w_x is related to the temporal frequency f by

$$w_x = \frac{f}{v} . \qquad \text{(A-4)}$$

Substituting (A-4) into (A-3) we obtain

$$f_2(vt,y) = \frac{1}{v} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} A_1(\frac{f}{v},w_y) F_1(\frac{f}{v},w_y) \exp[2\pi j(ft + w_y y)] df dw_y \qquad \text{(A-5)}$$

or

$$f_2(vt,y) = \int_{-\infty}^{\infty} \left[\frac{1}{v} \int_{-\infty}^{\infty} A_1(\frac{f}{v},w_y) F_1(\frac{f}{v},w_y) \exp(2\pi j w_y y) dw_y \right] \exp(2\pi j f t) df . \qquad \text{(A-6)}$$

Hence, the time-Fourier transform of the signal as seen by the detector is

$$F_2(f/y) = \frac{1}{v} \int_{-\infty}^{\infty} A_1(\frac{f}{v},w_y)F_1(\frac{f}{v},w_y)\exp(2\pi j w_y y)dw_y \qquad \text{(A-6)}$$

where f/y denotes the temporal frequency corresponding to a given scan line y. As shown in Figure A, the detector is followed by an electrical filter with a one-dimensional transfer function $A_2(f)$ to correct various system effects. The output voltage of this filter is

$$f_3(t/y) = \int_{-\infty}^{\infty} A_2(f)F_2(f/y)e^{2\pi j f t}df \quad . \qquad \text{(A-7)}$$

or

$$f_3(t/y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A_2(vw_x)A_1(w_x,w_y)F_1(w_x,w_y)\exp[2\pi j(xw_x+yw_y)]dw_x dw_y$$

and is dependent on the y coordinate of the line scan. One method of proceeding from this point is to assume that all scan lines are equivalent. This assumption suppresses the y coordinate and results in a time function $f_3(t)$ given by

$$f_3(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A_2(vw_x)A_1(w_x,w_y)F_1(w_x,w_y)\exp(2\pi j v t w_x)dw_x dw_y \quad . \qquad \text{(A-8)*}$$

The limitation of this procedure is that the output signal is one-dimensional, thereby prohibiting a true evaluation of system performance since the input signal is inherently two-dimensional. Furthermore, an analysis of this type permits only one-dimensional post filtering.

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*Other assumptions may lead to the same result.