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TITLE- A Statistical Analysis of a Bandpass Nonlinearity - Phase Detector Cascade

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ABSTRACT

In a previous memorandum (TM 67-2034-7) an approximate average power spectrum was derived for the response of an ideal limiter - phase detector - video filter cascade as a function of the signal-to-noise power ratio (SNR) at the limiter input. The input to the limiter was assumed to be the sum of a phase modulated sinusoid and a stationary gaussian noise. A complete solution is given in the present work for the response spectrum of a bandpass-zero memory - nonlinearity in cascade with a phase detector. The output power spectrum of any bandpass filter following this cascade is determined. In particular, the SNR can be found at the output of any filter that passes the power spectrum in a zone around a harmonic of the signal carrier frequency. When a video filter follows the phase detector and the nonlinearity is the ideal limiter, the complete solution is obtained for the problem considered in TM-67-2034-7. Some examples are presented of modulations for which the phase detector response has an average power spectrum. These are (1) biphase, (2) sinusoid with uniformly distributed random initial phase, and (3) the stationary gaussian process.

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FROM: W. D. Wynn

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TECHNICAL MEMORANDUM

INTRODUCTION

The mathematical model used to predict communication performance for the Apollo Unified S-Band System was defined by Mr. J. D. Hill in the Bellcomm Technical Memorandum TM #65-2021-3. Portions of this model have been questioned because of the lack of correlation between laboratory measurements and the predicted performance generated from the model. An area of the model that is frequently questioned is the treatment of the performance of the bandpass limiter - phase detector video filter cascade shown in Figure (1). This cascade exists in both the ground and spacecraft receivers of the Apollo USB System.

An approximate solution is known for the output power spectrum $S_z(\omega)$ of the cascade in Figure (1) as a function of the signal-to-noise power ratio (SNR) into the limiter.[(1)] This derivation of $S_z(\omega)$ holds for the signal

$$s(t) = P\cos[\omega_c t + \phi_c + \theta(t)] \tag{1}$$

where P and ω_c are constants, $\theta(t)$ is a member of a class of random process phase modulations, and ϕ_c is a random variable

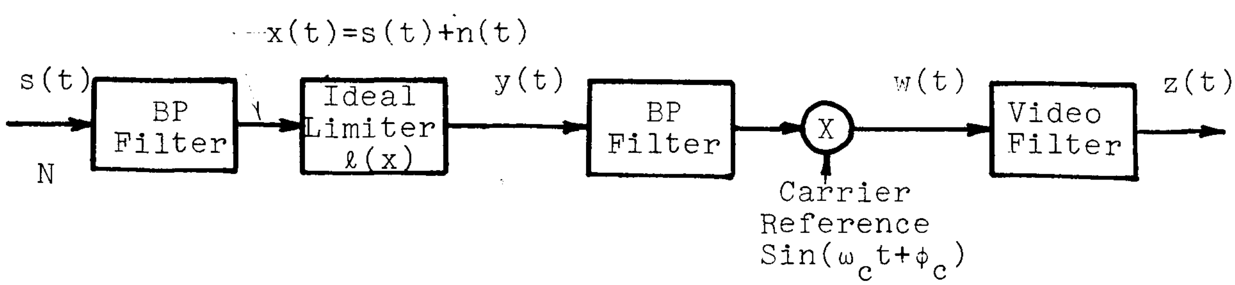

Figure 1. Cascade of A Bandpass Limiter - Phase Detector - Video Filter

representing the initial carrier phase. The carrier reference ϕ_c is assumed to be equally probable in the interval $[0,2\pi]$ with probability density $1/2\pi$. The signals considered are narrow band limited processes. The bandpass filters are correspondingly narrow bandpass around ω_c, and are symmetrical about ω_c. With these restrictions the post-limiter bandpass filter is superfluous in deriving $S_z(\omega)$.

In this memorandum the complete solution for $S_z(\omega)$ in reference 1 is obtained as a special case of a more general system analysis. A solution is given for the complete power spectrum of w(t) in Figure 2 as a function of the SNR into the zero memory nonlinearity g(x) when the signal is s(t) in equation (1) and n(t) is a stationary gaussian process with zero mean and variance σ^2.

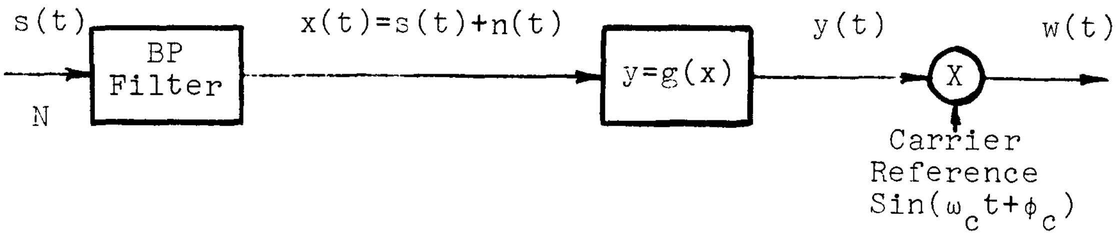

Figure 2. Cascade of Bandpass Nonlinearity and Phase Detector

For the cascade in Figure 1 the nonlinearity is

$$g(x) = \ell(x) = \begin{cases} +\ell \text{ when } x>0 \\ 0 \text{ when } x=0 \\ -\ell \text{ when } x<0 \end{cases} \quad (2)$$

where x = s+n.

The spectrum $S_w(\omega)$ is found as the Fourier transform of the autocorrelation function $R_w(t_1,t_2)$ when the latter is a function only of the time difference $\tau = t_2-t_1$. The time independence of R_w is a function of $\theta(t)$. As in the derivation of $S_z(\omega)$ in reference (1), there are some important cases of $\theta(t)$ for which $S_w(\omega)$ exists.

If the nonlinearity has a narrow band-limited input $x(t)$, the spectrum $S_w(\omega)$ is limited to narrow bands around the frequencies $\pm p\omega_c$ where $p=0,1,2,\ldots$. Then $S_z(\omega)$ is the part of $S_w(\omega)$ corresponding to $p=0$ when $g(x) = \ell(x)$ in Figure (2).

DISCUSSION

Derivation of $R_w(\tau)$ and $S_z(\omega)$

The analysis is based on the transform representation of a zero-memory nonlinearity.[2] A transform pair is defined by

$$f(\omega) = \int_{-\infty}^{+\infty} g(x)e^{-\omega x}dx$$

$$g(x) = \frac{1}{2\pi j} \int_C f(\omega)\, e^{x\omega}\, d\omega \tag{3}$$

where C is a contour in the complex ω-plane. In terms of the transform $f(\omega)$,

$$w(t) = \frac{1}{2\pi j} \int_C f(\omega) \sin(\omega_c t+\phi_c)e^{s\omega} \cdot e^{n\omega}\, d\omega \tag{4}$$

By definition

$$R_w(t_1,t_2) = E\,\{w(t_1) \cdot w(t_2)\} \tag{5}$$

where E{ } denotes the expectation taken with respect to ϕ_c,

$\theta(t)$ and $n(t)$. Throughout the development, it is assumed that θ, n and ϕ_c are jointly statistically independent.

For the symmetrical filter about ω_c the autocorrelation function of $n(t)$ is $R_n(\tau) = R_\nu(\tau) \cdot \cos \omega_c \tau$ where $\nu(t)$ is a low pass random function. Since $n(t)$ is stationary gaussian with variance $R_\nu(0) = \sigma^2$,

$$E\{\exp n(t_1)\omega_1 \cdot \exp n(t_2)\omega_2\} = \exp\left[\frac{\sigma^2\omega_1^2}{2} + R_\nu(\tau) \cdot \cos \omega_c\tau \cdot \omega_1\omega_2 + \frac{\sigma^2\omega_2^2}{2}\right] \tag{6}$$

where $E\{\ \}$ is the expectation with respect to n.(3)

With the substitution of (4) and (6) into (5) the identity(4)

$$\exp[a \cos z] = \sum_{m=0}^{\infty} \varepsilon_m I_m(a) \cos mz \tag{7}$$

$$\varepsilon_m = \begin{cases} 1, & m=0 \\ 2, & m=1,2,3, \ldots \end{cases}$$

gives $R_w(t_1,t_2)$. The complete result is

$$R_w(t_1,t_2) = \frac{1}{2}\sum_{q=0}^{\infty} \frac{R_\nu^{2q}\, h_{1,2q}^2}{2^{2q}(q!)^2} E\{\sin\theta_1 \cdot \sin\theta_2\} \tag{8}$$

$$+ \frac{1}{2}\sum_{p=1}^{\infty}\sum_{m=0}^{\infty}\sum_{q=0}^{\infty} \frac{R_\nu^u}{2^u\ q!(q+m)!}\Big[h_{\alpha,u}^2 E\{\cos[\alpha\theta_1-\alpha\theta_2+p\delta]\}$$

$$+ h_{\beta,u}^2 E\{\cos[\beta\theta_1-\beta\theta_2+p\delta]\}$$

$$- h_{\beta,u}h_{\alpha,u} E\{\cos[\beta\theta_1-\alpha\theta_2+p\delta]$$

$$+ \cos[\alpha\theta_1-\beta\theta_2+p\delta]\}\Big]$$

$$+ \frac{1}{2}\sum_{p=0}^{\infty}\sum_{m=1}^{\infty}\sum_{q=0}^{\infty} \frac{R_\nu^u}{2^u\ q!(q+m)!}\Big[h_{\gamma,u}^2 E\{\cos[\gamma\theta_1-\gamma\theta_2-p\delta]\}$$

$$+ h_{\xi,u}^2 E\{\cos[\xi\theta_1-\xi\theta_2-p\delta]\}$$

$$- h_{\xi,u}h_{\gamma,u} E\{\cos[\xi\theta_1-\gamma\theta_2-p\delta]$$

$$+ \cos[\gamma\theta_1-\gamma\theta_2-p\delta]\}\Big]$$

where $\delta = \omega_c\tau$

$u = 2q + m$

$\alpha = m - p + 1$

$\beta = m - p - 1$

$\gamma = m + p + 1$

$\xi = m + p - 1$

For each pair of integers (r,k), $h_{r,k}$ is a constant given by

$$h_{r,k} = \frac{1}{2\Pi j} \int_C f(\omega) e^{\frac{\sigma^2 \omega^2}{2}} \omega^k I_r(\omega P) d\omega \qquad (9)$$

The details of the derivation of (8) are in Appendix A-I.

In equation (8) there are three key factors. These are:

1. $R_\nu(\tau)$, the autocorrelation function of the lowpass equivalent process for n(t),
2. $h_{r,k}$, a constant for each pair of integers (r,k) that is determined by the form of g(x) and the input SNR to the nonlinearity, and
3. the expectation with respect to θ

$$E\{\cos[A\theta_1 - B\theta_2 \pm p\delta]\} \qquad (10)$$

where A, B and p are integers. Since R_ν is independent of time, the function $R_W(t_1,t_2)$ depends only on τ if (10) has this property.

Some important cases of $\theta(t)$ for which (10) is time independent are listed below. Equation (10) is derived for each case in Appendix II.

1. The biphase waveform $\theta(t) = \pm|\theta|$ where the mean is zero and the auto correlation function is[5]

$$R_\theta(\tau) = |\theta|^2 (1 - \frac{|\tau|}{T}) \quad \text{for } |\tau| \leq T \qquad (11)$$

and zero for all $|\tau| > T$. For this case (10) is given by (A-20).

2. The random tone

$$\theta(t) = m_1 \sin(\omega_1 t+\xi) \tag{12}$$

where ξ is a random variable with a uniform probability density on $[0,2\Pi]$. In this case (10) is given in (A-21).

3. The stationary gaussian process with zero mean and variance σ_θ^2. For this modulation (10) is given in (A-24).

When (10) depends only on τ for integers A, B and p, $S_w(\omega)$ is the infinite sum of terms of the type

$$F[R_\nu^u] * F[E\{\cos[A\theta_1-B\theta_2 \pm p\delta]\}] \tag{13}$$

where F denotes the Fourier transform with respect to τ and (*) is the convolution of the two frequency spectra.

In Appendix A-II it is shown that $E\{\cos[A\theta_1-B\theta_2]\}$ depends only on τ and that $E\{\sin[A\theta_1-B\theta_2]\}$ is zero for the three modulation cases presented above. For each case, $S_w(\omega)$ is given by (A-25). The spectrum is an infinite sum of terms with the form

$$F[R_\nu^u(\tau)] * F[\cos p\omega_c\tau] * S_\theta(A,B,\omega) \tag{14}$$

where

$$S_\theta[A,B,\omega] = F[E\{\cos[A\theta_1-B\theta_2]\}]$$

for any pair of integers A and B.

In (13) the spectrum $F[R_\nu^u]$ is a result of the noise n(t). Then any term of the form (13) in (8) with $u \neq 0$ is an interference term. For the first sum in (8), the $R_\nu^{(0)}$ term is

$$\frac{1}{2} h_{1,0}^2 \; E\{\sin\theta_1 \sin\theta_2\} \tag{15}$$

For the second sum in (8), the $R_\nu^{(0)}$ terms occur when $2q+m = 0$ where $q \geq 0$ and $m \geq 0$. These terms are

$$\frac{1}{2} \sum_{p=1}^{\infty} h_{p-1,0}^2 \; E\{\cos[(p-1)\theta_1-(p-1)\theta_2-p\delta]\} +$$

$$h_{p+1,0}^2 \; E\{\cos[(p+1)\theta_1-(p+1)\theta_2-p\delta]\} -$$

$$h_{p+1,0} h_{p-1,0} \; E\{\cos[(p+1)\theta_1-(p-1)\theta_2-p\delta]$$

$$+ \cos[(p-1)\theta_1-(p+1)\theta_2-p\delta]\} \tag{16}$$

The term (15) and some of those in (16) are "signal" terms. Not all terms of (16) can be called signal terms since some of these are intermodulation. The signal term (15) appears at the output of the video filter in Figure (1). The signal components in (16) exist around integer multiples of the carrier ω_c and can be obtained with a bandpass filter instead of the video filter following the phase detector in Figure (1).

The function $R_z(t_1,t_2)$ is found for the system in Figure (1) by taking only the p=0 term of equation (8). When θ is such that $R_z(t_1,t_2) = R_z(\tau)$ the complete spectrum of z is

$$S_z(\omega) = \frac{1}{4} \sum_{m=0}^{\infty} \sum_{q=0}^{\infty} \frac{\varepsilon_m}{2^u \; q! \; (q+m)!} \cdot \tag{17}$$

$$\Big[h_{a,u}^2 \; S_\nu(\omega,u) * S_\theta(a,a,\omega) + h_{b,u}^2 \; S_\nu(\omega,u) * S_\theta(b,b,\omega)$$

$$- h_{a,u} \, h_{b,u} \; S_\nu(\omega,u) * \{S_\theta(a,b,\omega) + S_\theta(b,a,\omega)\} \Big]$$

where $a = m+1$, $b = m-1$, $u = 2q+m$,

$$S_\nu(\omega,u) = \mathcal{F}[R_\nu^u(\tau)]$$

and

$$S_\theta(a,b,\omega) = \mathcal{F}[E\{\cos[a\theta_1 - b\theta_2]\}]$$

The h-Parameters When $g(x) = \ell(x)$

When $g(x)$ is the ideal limiter, equation (9) has a closed form solution in terms of the modified Bessel functions $I_0(x/2)$ and $I_1(x/2)$ where $x = P^2/2\sigma^2$ is the input SNR to the limiter.[6] In Table 1, the lower order h parameters are given. From the entries in Table 1 any h parameter in equation (8) can be found with the recursion relationships A-30, A-31 and A-32.

r	k	$h_{r,k}$
1	0	$\frac{\ell P}{\sqrt{2\pi}\sigma} e^{-x/2} \left[I_0(x/2) + I_1(x/2)\right]$
0	1	$\frac{\sqrt{2}\ \ell}{\sqrt{\pi}\ \sigma} e^{-x/2} I_0(x/2)$
2	1	$\frac{-\sqrt{2}\ell}{\sqrt{\pi}\ \sigma} e^{-x/2} I_1(x/2)$
1	2	$\frac{-\ell P}{\sqrt{2\pi}\sigma^3} e^{-x/2} \left[I_0(x/2) - I_1(x/2)\right]$
3	2	$\frac{-\ell P}{\sqrt{2\pi}\sigma^3} e^{-x/2} \left[I_0(x/2) - \left(1 + \frac{4}{x}\right) I_1(x/2)\right]$
0	3	$\frac{-\sqrt{2}\ell}{\sqrt{\pi}\sigma^3} e^{-x/2} \left[(1-x)I_0(x/2) + xI_1(x/2)\right]$
2	3	$\frac{\ell P^2}{\sqrt{2\pi}\sigma^5} e^{-x/2} \left[I_0(x/2) - \left(1 + \frac{1}{x}\right) I_1(x/2)\right]$
4	3	$\frac{-\ell\ P^2}{\sqrt{2\pi}\sigma^5} e^{-x/2} \left[\left(1 + \frac{4}{x} + \frac{12}{x^2}\right) I_1(x/2) - \left(1 + \frac{3}{x}\right) I_0(x/2)\right]$
1	4	$\frac{\ell P}{\sqrt{2\pi}\sigma^5} e^{-x/2} \left[(3-2x)I_0(x/2) + (2x-1)I_1(x/2)\right]$

Table 1. Closed Form Solutions of $h_{r,k}$

Conclusions and Remarks

A general second order statistical analysis is presented for the cascade of a narrow bandpass nonlinearity and an ideal phase detector. In this analysis, the input to the nonlinearity is assumed to be the sum of a stationary gaussian noise and a fixed amplitude phase modulated sine wave. The autocorrelation function of the cascade response is obtained as a function of the signal-to-noise ratio x at the nonlinearity input, the normalized autocorrelation function of the lowpass equivalent for the nonlinearity input noise, $r_v(\tau)$; and the phase modulation $\theta(t)$.

In general, the cascade response w(t) has the autocorrection function $R_w(t_1,t_2)$ that is time dependent. However, for some important cases of $\theta(t)$, $R_w(t_1,t_2) = R_w(\tau)$, and the cascade response has the average power spectrum $S_w(\omega) = F[R_w(\tau)]$ where F is the Fourier transform operation with respect to τ. The cases of $\theta(t)$ considered that yield $R_w(\tau)$ are the random biphase waveform $\theta = \pm|\theta|$, the single tone $\theta(t) = m_1 \sin(\omega_1 t+\xi)$, and the stationary gaussian process with autocorrelation function $K_\theta(\tau)$ and zero mean.

The dependence of $R_w(t_1,t_2)$ on the nonlinearity input SNR x appears in the h parameters. These parameters can be obtained in closed form as functions of the modified Bessel functions $I_0(\frac{x}{2})$ and $I_1(\frac{x}{2})$. The lower order h parameters encountered in the first few terms of the series for R_w were found, and recurrence relations were derived with which higher order h parameters follow easily.

For the modulation types that make R_w a function of τ alone the power spectrum $S_w(\omega)$ is known for all values of the input SNR x into the nonlinearity. This spectrum consists of spectral zones around $n\omega_c$ where n = 0, 1, 2, ... In each zone there are three types of terms; signal noise, and intermodulation. In any frequency band at the output of the phase detector in Figure 2, the SNR can be determined for any value of x into the nonlinearity. The spectrum $S_z(w)$ for Figure (1) is just the low-pass part of $S_w(w)$ corresponding to n=0 The complete spectra for w(t) and z(t) are obtained in the present work as infinite series, but in any numerical calculations such as the computation of output SNR, it is necessary to limit the infinite series representations for S_w or S_z to a finite number of terms. Usually only a modest number of terms for S_w or S_z are necessary to give a good approximation to either spectrum in any finite output frequency bandwidth.

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2034-WDW-jr

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Attachments
Appendices I-III

APPENDIX I

DERIVATION OF $R_w(t_1,t_2)$ IN EQUATION (8)

For the analysis, the input x(t) to the nonlinearity in Figure (2) is assumed to be narrowband limited by the symmetrical bandpass filter with center frequency ω_c. The Laplace transform solution of a zero memory nonlinearity with stochastic excitation is used to derive $R_w(t_1,t_2)$.[7] The nonlinearity g(x) is given by

$$g(x) = \frac{1}{2\pi j}\left[\int_{C_+} f_+(\omega)e^{x\omega}d\omega + \int_{C_-} f_-(\omega)e^{x\omega}d\omega\right] \qquad (A-1)$$

where

$$f_+(\omega) = \int_0^{+\infty} g(x)e^{-\omega x}dx \quad \text{for } Re[\omega] > 0$$

and

$$f_-(\omega) = \int_{-\infty}^{0} g(x)e^{-\omega x}dx \quad \text{for } Re[\omega] < 0$$

The variable $\omega = u + jv$ is complex with $Re[\omega] = u$. The contours C_+ and C_- are taken parallel to the v-axis in the ω-plane with $Re[\omega] > 0$ for C_+ and $Re[\omega] < 0$ for C_-. For convenience (A-1) is written as

$$g(x) = \frac{1}{2\pi j}\int_C f(\omega)e^{x\omega}d\omega \qquad (A-2)$$

Since $w(t) = \sin(\omega_c t + \phi) \cdot g(x(t))$, the autocorrelation function of $w(t)$ is

$$R(t_1,t_2) = \left(\frac{1}{2\pi j}\right)^2 \int_C f(\omega_1) \int_C f(\omega_2) \, E\left\{\sin(\omega_c t_1 + \phi) \cdot e^{\omega_1 s_1 + \omega_1 n_1} \right.$$

$$\left. \cdot \sin(\omega_c t_2 + \phi) \cdot e^{\omega_2 s_2 + \omega_2 n_2}\right\} d\omega_1 d\omega_2 \quad \text{(A-3)}$$

where $s_i = s(t_i)$ and $n_i = n(t_i)$, $i=1,2$. The order of complex integration and the expectation operation have been interchanged to get (A-3). For the assumed statistical independence of $n(t)$, $\theta(t)$ and ϕ, the expected value in (A-3) factors into

$$E\left\{\sin(\omega_c t_1 + \phi) \cdot e^{\omega_1 s_1} \cdot \sin(\omega_c t_2 + \phi) \cdot e^{\omega_2 s_2}\right\} \cdot$$

$$\exp \tfrac{1}{2}[\sigma^2\omega_1^2 + 2R_n(\tau)\omega_1\omega_2 + \sigma^2\omega_2^2] \quad \text{(A-4)}$$

where $\tau = t_2 - t_1$. The form for the cross correlation function $E\left\{e^{\omega_1 n_1} e^{\omega_2 n_2}\right\}$ where $n(t)$ is stationary gaussian[3] has been used in (A-4).

For the case where $s(t)$ is narrow band-limited with respect to ω_c, the filter in Figure (2) is narrow bandpass and is assumed to be symmetrical about ω_c. Then $n(t)$ can be written as[8]

$$n(t) = x_c \cos \omega_c t - x_s \sin \omega_c t \quad \text{(A-5)}$$

where x_c and x_s are statistically independent stationary gaussian random processes; and

$$R_n(\tau) = R_\nu(\tau) \cos \omega_c \tau \qquad \text{(A-6)}$$

where $R_\nu(\tau) = R_{x_c}(\tau) = R_{x_s}(\tau)$. For a narrow bandpass IF filter the transform of $R_\nu(\tau)$ is lowpass with a narrow bandwidth compared to ω_c.

With the substitution of

$$t_1 = t \quad , t_2 = t + \tau$$

$$\phi^* = \phi + \omega_c t$$

$$\sin \phi^* = \frac{e^{j\phi^*} - e^{-j\phi^*}}{2j}$$

and[4]

$$e^{R_n(\tau)\omega_1\omega_2} = \sum_{m=-\infty}^{+\infty} I_m(\omega_1\omega_2 R_\nu)e^{jm\omega_c\tau} \qquad \text{(A-7)}$$

(A-3) becomes

$$R_\omega(t_1,t_2) = \frac{1}{(2\Pi j)^2} \int_C \int_C f(\omega_1)f(\omega_2)e^{\frac{\sigma^2\omega_1^2}{2}} e^{\frac{\sigma^2\omega_2^2}{2}} \left(-\frac{1}{4}\right) \sum_{m=-\infty}^{+\infty} I_m(\omega_1\omega_2 R_\nu)$$

$$\cdot\ h(\delta,\omega_1,\omega_2)d\omega_1 d\omega_2 \qquad \text{where}$$

$$h(\delta,\omega_1,\omega_2) = e^{jm\delta} \cdot E\{[e^{j\delta+j2\phi^*}+e^{-j\delta-j2\phi^*}-e^{j\delta}-e^{-j\delta}] \cdot$$
$$\cdot\ e^{\omega_1 P\cos(\theta_1+\phi^*) + \omega_2 P\cos(\theta_2+\phi^*+\delta)}\} \tag{A-8}$$

Here $\delta = \omega_2\tau$, and the expectation E is with respect to ϕ^* and θ.

The function $h(\delta,\omega_1,\omega_2)$ is periodic in δ with period 2Π. The exponential Fourier series representation of h is

$$h(\delta,\omega_1,\omega_2) = \sum_{p=-\infty}^{p=+\infty} \alpha_p e^{jp\delta} \qquad \text{where}$$

$$\alpha_p = \frac{1}{2\Pi} \int_{-\Pi}^{+\Pi} h(\delta,\omega_1\omega_2)\ e^{-jp\delta}\ d\delta\ . \tag{A-9}$$

Then

$$\alpha_p = \frac{1}{2\Pi} \int_{-\Pi}^{+\Pi} e^{j(m-p)\delta}\ E\{\quad\}\ d\delta\ ,$$

and since ϕ^* has a uniform probability density function on $[0,2\pi]$ if ϕ has this density function,

$$\alpha_p = \sum_{r=-\infty}^{+\infty} \sum_{k=-\infty}^{+\infty} I_r(\omega_1 P) I_k(\omega_2 P) \cdot$$

$$E_\theta \left\{ \int_{-\Pi}^{\Pi} \frac{d\delta}{2\Pi} \int_{-\Pi}^{\Pi} \frac{d\phi^*}{2\Pi} \left[e^{j(m+1-p+k)\delta + j(2+r+k)\phi^* + j(r\theta_1 + k\theta_2)} \right.\right.$$
$$+ e^{j(m-p-1+k)\delta + j(-2+r+k)\phi^* + j(r\theta_1 + k\theta_2)}$$
$$- e^{j(m-p+1+k)\delta + j(r+k)\phi^* + j(r\theta_1 + k\theta_2)}$$
$$\left.\left. - e^{j(m-p-1+k)\delta + j(r+k)\phi^* + j(r\theta_1 + k\theta_2)} \right] \right\} \qquad \text{(A-10)}$$

There are four terms to consider in (A-10). The first term integrates to zero unless

$$k = p - m - 1 \quad \underline{\text{and}} \quad r = m - p - 1$$

The second term is zero unless

$$k = -m + 1 + p \quad \underline{\text{and}} \quad r = m + 1 - p$$

The third term is zero unless

$$k = p - m - 1 \quad \underline{\text{and}} \quad r = -p + m + 1$$

Finally the fourth term is zero unless

$$k = p - m + 1 \quad \underline{\text{and}} \quad r = m - p - 1$$

Then in (A-8)

$$\left(-\frac{1}{4}\right) \sum_{m=-\infty}^{+\infty} I_m(\omega_1 \omega_2 R_\nu) h(\delta, \omega_1, \omega_2) =$$

$$\left(-\frac{1}{4}\right) \sum_{p=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} I_m(\omega_1 \omega_2 R_\nu) \alpha_p e^{jp\delta}$$

$$= \left(-\frac{1}{4}\right) \sum_{p=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} \sum_{r=-\infty}^{+\infty} \sum_{k=-\infty}^{+\infty} I_m(\omega_1 \omega_2 R_\nu) I_r(\omega_1 P) I_k(\omega_2 P) e^{jp\delta} E_\theta \{ \quad \}$$

$$= \left(-\frac{1}{4}\right) \sum_{p=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} e^{jp\delta} I_m(\omega_1 \omega_2 R_\nu) \quad \cdot$$

$$\Big[I_{m-1-p,1} I_{-m-1+p,2}\, E\left\{ e^{j(m-1-p)\theta_1 + j(-m-1+p)\theta_2} \right\}$$

$$+ I_{m+1-p,1} I_{-m+1+p,2}\, E\left\{ e^{j(m+1-p)\theta_1 + j(-m+1+p)\theta_2} \right\}$$

$$- I_{m+1-p,1} I_{-m-1+p,2}\, E\left\{ e^{j(m+1-p)\theta_1 + j(-m-1+p)\theta_2} \right\}$$

$$- I_{m-1-p,1} I_{-m+1+p,2}\, E\left\{ e^{j(m-1-p)\theta_1 + j(-m+1+p)\theta_2} \right\} \Big] \qquad \text{(A-11)}$$

where $I_{n,1} = I_n(\omega_1 P)$ and $I_{n,2} = I_n(\omega_2 P)$.

Considering the composition of the double sum in equation (A-11),

$$\sum_{p=-\infty}^{+\infty}\sum_{m=-\infty}^{+\infty} = \sum_{p=1}^{+\infty}\sum_{m=1}^{+\infty} + \sum_{p=1}^{+\infty}\sum_{m=-\infty}^{-1} + \sum_{p=-\infty}^{-1}\sum_{m=1}^{+\infty} + \sum_{p=-\infty}^{-1}\sum_{m=-\infty}^{-1}$$

$$+ \sum_{p=0}\sum_{m=1}^{+\infty} + \sum_{p=0}\sum_{m=-\infty}^{-1} + \sum_{p=1}^{+\infty}\sum_{m=0} + \sum_{p=-\infty}^{-1}\sum_{m=0} + \sum_{p=0}\sum_{m=0} \qquad \text{(A-12)}$$

where $\sum_{p=0}\sum_{m=0}$ denotes the term corresponding to p=0 and m=0. Since $I_{-k} = I_{+k}$ the double sum corresponding to (p>0, m>0) can be combined with the double sum corresponding to (p<0, m>0). Similarly, the double sum corresponding to (p<0, m>0) can be combined with the double sum corresponding to (p>0, m<0). The combination of the series for (p>0, m>0) and (p<0, m<0) gives

$$\left(-\frac{1}{2}\right)\sum_{p=+1}^{+\infty}\sum_{m=+1}^{+\infty} I_m(\omega_1\omega_2R_\nu)\Big[I_{\beta,1}\cdot I_{\alpha,2}\cdot E\{\cos[\beta\theta_1-\alpha\theta_2+p\delta]\}$$

$$+ I_{\alpha,1}\cdot I_{\beta,2}\cdot E\{\cos[\alpha\theta_1-\beta\theta_2+p\delta]\}$$

$$- I_{\alpha,1}\cdot I_{\alpha,2}\cdot E\{\cos[\alpha\theta_1-\alpha\theta_2+p\delta]\}$$

$$- I_{\beta,1}\cdot I_{\beta,2}\cdot E\{\cos[\beta\theta_1-\beta\theta_2+p\delta]\}\Big] \qquad \text{(A-13)}$$

where $\alpha = m - p + 1$ and $\beta = m - p - 1$. The combination of the series for (p<0, m>0) and (p>0, m<0) gives

$$\left(-\frac{1}{2}\right) \sum_{p=+1}^{+\infty} \sum_{m=+1}^{+\infty} I_m(\omega_1\omega_2R_\nu) \cdot \Big[I_{\xi,1} \cdot I_{\gamma,2} \cdot E\{\cos[\xi\theta_1-\gamma\theta_2-p\delta]\}$$

$$+ I_{\gamma,1} \cdot I_{\xi,2} \cdot E\{\cos[\gamma\theta_1-\xi\theta_2-p\delta]\}$$

$$- I_{\gamma,1} \cdot I_{\gamma,2} \cdot E\{\cos[\gamma\theta_1-\gamma\theta_2-p\delta]\}$$

$$- I_{\xi,1} \cdot I_{\xi,2} \cdot E\{\cos[\xi\theta_1-\xi\theta_2-p\delta]\}\Big] \qquad \text{(A-14)}$$

where $\gamma = m + p + 1$ and $\xi = m + p - 1$. The combination of the series for (p=0, m>0) and (p=0, m<0) gives

$$\left(-\frac{1}{2}\right) \sum_{m=+1}^{+\infty} I_m(\omega_1\omega_2R_\nu) \cdot \Big[I_{m-1,1} \cdot I_{m+1,2} \cdot E\{\cos[(m-1)\theta_1-(m+1)\theta_2]\}$$

$$+ I_{m+1,1} \cdot I_{m-1,2} \cdot E\{\cos[(m+1)\theta_1-(m-1)\theta_2]\}$$

$$- I_{m+1,1} \cdot I_{m+1,2} \cdot E\{\cos[(m+1)\theta_1-(m+1)\theta_2]\}$$

$$- I_{m-1,1} \cdot I_{m-1,2} \cdot E\{\cos[(m-1)\theta_1-(m-1)\theta_2]\}\Big] \qquad \text{(A-15)}$$

The term corresponding to (m=0, p=0) is

$$+ \frac{1}{2} I_0(\omega_1\omega_2 R_\nu) I_{1,1} \; I_{1,2} \; E\{\sin\theta_1 \sin\theta_2\} \tag{A-16}$$

The combination of the series for (m=0, p>0) and (m=0, p<0) gives

$$\begin{aligned}\left(-\frac{1}{2}\right) \sum_{p=+1}^{+\infty} I_0(\omega_1\omega_2 R_\nu) \cdot \Big[& I_{p+1,1} I_{p-1,2} E\{\cos[(p+1)\theta_1-(p-1)\theta_2-p\delta]\} \\ & + I_{p-1,1} I_{p+1,2} E\{\cos[(p-1)\theta_1-(p+1)\theta_2-p\delta]\} \\ & - I_{p-1,1} I_{p-1,2} E\{\cos[(p-1)\theta_1-(p-1)\theta_2-p\delta]\} \\ & - I_{p+1,1} I_{p+1,2} E\{\cos[(p+1)\theta_1-(p+1)\theta_2-p\delta]\} \Big]\end{aligned} \tag{A-17}$$

When (A-13) through (A-17) along with[9]

$$I_m(\omega_1\omega_2 R_\nu) = \sum_{q=0}^{+\infty} \frac{\omega_1^{m+2q} \; \omega_2^{m+2q} \; R_\nu^{m+2q}}{2^{m+2q} \; q! \; (q+m)!} \tag{A-18}$$

and

$$h_{r,k} = \frac{1}{2\pi j} \int_C f(\omega) e^{\frac{\sigma^2 \omega^2}{2}} \omega^k I_r(\omega P) d\omega \tag{9}$$

are substituted in (A-8), equation (8) is obtained for $R_W(t_1,t_2)$.

APPENDIX II

DERIVATION OF $E\{\cos[A\theta_1 - B\theta_2 \pm p\delta]\}$ FOR SOME CASES OF $\theta(t)$

Consider $\theta(t) = \pm|\theta|$, a biphase waveform with zero mean value and the autocorrelation function given in equation (11). Equation (10) is

$$E\{\cos[A\theta_1 - B\theta_2 \pm p\delta]\}$$

$$= \cos p\,\delta \cdot E\{\cos[A\theta_1 - B\theta_2]\} \mp \sin p\,\delta \cdot E\{\sin[A\theta_1 - B\theta_2]\}$$

$$= \cos p\,\delta \cdot E\{\cos A\theta_1 \cdot \cos B\theta_2\}$$

$$+ \cos p\,\delta \cdot E\{\sin A\theta_1 \cdot \sin B\theta_2\}$$

$$\mp \sin p\,\delta \cdot E\{\sin A\theta_1 \cdot \cos B\theta_2\}$$

$$\pm \sin p\,\delta \cdot E\{\cos A\theta_1 \cdot \sin B\theta_2\} \qquad \text{(A-19)}$$

But when $\theta(t) = \pm|\theta|$, $\cos A\,\theta = \cos A|\theta|$, a constant. Also $\sin A\,\theta = \frac{\theta}{|\theta|} \sin A\,|\theta|$. Then since $E\{\theta\} = 0$, (10) reduces to

$$\cos p\,\delta \cdot E\{\cos[A\theta_1 - B\theta_2]\} = \qquad \text{(A-20)}$$

$$\cos p\,\delta \cdot [\cos A|\theta| \cdot \cos B|\theta| + \sin A|\theta| \cdot \sin B|\theta| \cdot r_\theta(\tau)]$$

where $r_\theta(\tau)$ is equation (11) normalized by $|\theta|^2$.

If $\theta(t) = m_1 \sin(\omega_1 t+\xi)$, where ξ is a random variable with uniform distribution, two terms combine in (A-19) to give[1]

$$\cos p\,\delta \cdot E\{\cos A\theta_1 \cdot \cos B\theta_2 + \sin A\theta_1 \cdot \sin B\theta_2\}$$

$$= \cos p\,\delta \cdot \sum_{n=0}^{+\infty} \varepsilon_n J_n(Am_1)\, J_n(Bm_1)\, \cos(n\omega_1\tau) \qquad \text{(A-21)}$$

Since

$$\cos[Am_1 \sin(\omega_1 t_1+\xi)] \cdot \sin[Bm_1 \sin(\omega_1 t_2+\xi)] =$$

$$2 \sum_{m=1}^{+\infty} \sum_{n=0}^{+\infty} \varepsilon_n J_{2n}(Am_1)\cos[2n(\omega_1 t_1+\xi)] \cdot$$

$$J_{2m-1}(Bm_1)\sin[(2m-1)(\omega_1 t_2+\xi)] \quad , \qquad \text{(A-22)}$$

and

$$E\{\cos[2n(\omega_1 t_1+\xi)] \cdot \sin[(2m-1)(\omega_1 t_2+\xi)]\} = 0$$

for any combination of $m \geq 1$ and $n \geq 0$; (A-19) reduces to (A-21) for the single tone modulation.

When $\theta(t)$ is a stationary gaussian process with zero mean and variance σ_θ^2, consider the second order characteristic equation for the gaussian process[10]

$$\Phi_\theta(A,-B,\tau) = E\left\{e^{jA\theta_1 - jB\theta_2}\right\} = \exp\left[-\frac{\sigma_\theta^2}{2}(A^2+B^2)+ABK_\theta(\tau)\right] \quad (A\text{-}23)$$

Since (A-23) is real, (A-19) reduces to

$$\cos p\,\delta \cdot \Phi_\theta(A,B,\tau) \quad (A\text{-}24)$$

By considering (A-20), (A-21) and (A-24) it is found that

$$E\{\cos[A\theta_1-B\theta_2]\} = E\{\cos[B\theta_1-A\theta_2]\}$$

Then equation (8) reduces to the simpler form

$$R_W(\tau) = \frac{1}{2}\sum_{q=0}^{+\infty}\left\{\frac{R_V^{2q}\;h_{1,2q}^2}{2^{2q}\;(q!)^2}\;E\{\sin\theta_1\cdot\sin\theta_2\}\right.$$

$$+\sum_{p=1}^{+\infty}\sum_{m=0}^{+\infty}\frac{R_\nu^u\cos p\,\delta}{2^u\;q!\;(q+m)!}\left[h_{\alpha,u}^2\,R_\theta(\alpha,\alpha,\tau)+h_{\beta,u}^2\,R_\theta(\beta,\beta,\tau)\right.$$

$$\left.-\,2\,h_{\alpha,u}h_{\beta,u}\,R_\theta(\alpha,\beta,\tau)\right]+\sum_{p=0}^{+\infty}\sum_{m=1}^{+\infty}\frac{R_\nu^u\cos p\,\delta}{2^u\;q!\;(q+m)!}\left[h_{\gamma,u}^2\,R_\theta(\gamma,\gamma,\tau)\right.$$

$$\left.\left.+\,h_{\xi,u}^2\,R_\theta(\xi,\xi,\tau)-2h_{\gamma,u}h_{\xi,u}\,R_\theta(\gamma,\xi,\tau)\right]\right\} \quad (A\text{-}25)$$

where $R_\theta(A,B,\tau) = E\{\cos\,[A\theta_1-B\theta_2]\}$.

APPENDIX III

THE CLOSED FORM SOLUTION FOR h_{mk} WHEN $g(x) = \ell(x)$

The autocorrelation function of w(t) given in (8) contains the constants h_{rk} where r+k are odd integers. For the ideal limiter characteristic (2), closed form solutions exist for these parameters. Since $f_+(\omega) = \ell/\omega$ for $Re[\omega] > 0$ and $f_-(\omega) = \ell/\omega$ for $Re[\omega] < 0$, (9) becomes

$$h_{r,k} = \frac{1}{2\pi j} \int_{C_-} \ell\omega^{k-1} I_r(\omega P) \exp\left(\frac{\sigma^2\omega^2}{2}\right) d\omega$$

$$+ \frac{1}{2\pi j} \int_{C_+} \ell\omega^{k-1} I_r(\omega P) \exp\left(\frac{\sigma^2\omega^2}{2}\right) d\omega \qquad \text{(A-26)}$$

where C_- is the contour $(-\varepsilon-j\infty, -\varepsilon+j\infty)$ and C_+ is the contour $(+\varepsilon-j\infty, +\varepsilon+j\infty)$. By the change of variable $\omega=jx$ and the substitution of $I_r(z) = (j)^{-r} J_r(jz)$, analytic continuation can be applied for $r\geq 0$ and $k\geq 0$ to give

$$h_{r,k} = \frac{\ell}{\pi} (j)^{k+r-1} \int_{-\infty}^{\infty} x^{(k-1)} J_r(xP) \exp\left[\frac{-\sigma^2 x^2}{2}\right] dx \qquad \text{(A-27)}$$

When r+k is even the integrand of (A-27) is odd and $h_{r,k} = 0$. When r+k is odd the integrand of (A-27) is even and

$$h_{r,k} = \frac{2\ell}{\pi} (j)^{k+r-1} \cdot \frac{\Gamma\left(\frac{r+k}{2}\right)\left(\frac{P^2}{2\sigma^2}\right)^{r/2}}{2\Gamma(r+1)\left(\frac{\sigma}{\sqrt{2}}\right)^k} \, {}_1F_1\left(\frac{r+k}{2} \; ; \; r+1 \; ; \; \frac{-P^2}{2\sigma^2}\right) \qquad \text{(A-28)}$$

where a solution has been used for the integral

$$\int_0^\infty x^{k-1} J_r(xP) \exp\left[\frac{-\sigma^2 x^2}{2}\right] dx \qquad \text{(A-29)}$$

in terms of the confluent hypergeometric function ${}_1F_1(\alpha,\beta,-x)$.(11) For the case when r and k are nonnegative integers ${}_1F_1\left(\frac{r+k}{2}; r+1; -x\right)$ can be expressed in closed form in terms of first and second kind modified Bessel functions. A list of these expressions is given by Middleton.(12) A collection of $h_{r,k}$ in closed form for low order indices is given in Table 1. For Table 1, $x = P^2/2\sigma^2$ is the input signal-to-noise power ratio into the limiter.

If $g(x) = \ell(x)$, any of the $h_{r,k}$ in (8) can be found in closed form from Table 1, by using the recurrence relations

$$h_{r+2,k} = h_{r,k} - \frac{2(r+1)}{P} h_{r-1,k-1} + \frac{4(r+1)r}{P^2} h_{r,k-2} \qquad \text{(A-30)}$$

$$h_{r+1,k+1} = -\frac{P}{\sigma^2} h_{r,k} - \frac{(k-r-2)}{\sigma^2} h_{r-1,k-1} + \frac{2(k-r-2)r}{\sigma^2 P} h_{r,k-2} \qquad \text{(A-31)}$$

and

$$h_{r,k+2} = \frac{(r-k)}{\sigma^2} h_{r,k} + \frac{P^2}{\sigma^4} h_{r-2,k} - \frac{(r-k)}{\sigma^4} P\, h_{r-1,k-1} \qquad \text{(A-32)}$$

Equation (A-30) is derived from (A-27) by using the Bessel function identity[13]

$$J_{r+2}(xP) = \frac{2(r+1)}{Px} J_{r+1}(xP) - J_r(xP) \quad \text{(A-33)}$$

Equation (A-31) is derived through a by-parts integration of (A-27) and the application of (A-30). Equation (A-32) is derived through by-parts integration of (A-27). In the development of (A-30), (A-31) and (A-32) the integral in (A-27) is restricted to $[0,\infty)$. This is possible since the integrand in (A-27) is even when r+k is odd.

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