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COVER SHEET FOR TECHNICAL MEMORANDUM

TITLE- A Simple and Efficient Decision Criterion for Operating in a Pulse Position Modulation System when both Signal and Noise are Poisson Distributed*

TM- 68-2034-12

DATE- July 10, 1968

FILING CASE NO(S)- 900

AUTHOR(S)- L. Schuchman

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ABSTRACT

In this paper an M'ary receiver decision logic is derived which requires no threshold. Thus the detector is not a function of the signal and noise energy parameters. In addition the performance of the M'ary system is derived and graphical results presented which illustrate the M'ary performance.

*The major results derived by the author have recently, and unknown to the author, been published in a paper entitled, "M-ary Poisson Detection and Optical Communications," by S. Karp and R. M. Gagliardi, NASA Tech. Note D-4623, June 1968. The authors approach differs from that of Karp and Gagliardi in that I was not attempting to prove optimality of the system. Indeed, it is shown that for M=2 this system is not optimum.

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SUBJECT: A Simple and Efficient Decision Criterion for Operating in a Pulse Position Modulation System when both Signal and Noise are Poisson Distributed
Case 900

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FROM: L. Schuchman

TM-68-2034-12

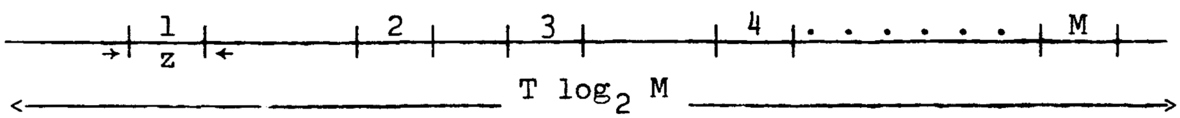
TECHNICAL MEMORANDUM

INTRODUCTION

Several researchers[1,2,3] have described the detection of optical signals as a counting of signal photons. The predominant noise in these models also is described in terms of photons whose arrival times and that of the signal are assumed to be statistically characterized by the Poisson process. Previously the performance of the optimum detector for a binary coded system was derived.[1,2] In this paper we extend this work and derive the performance of the optimum detector when the digital code is of an order greater than two. In addition, we have chosen a somewhat different hypothesis for the binary case which leads to a different, less optimum, but simpler receiver than Reiffen and Sherman.[1]

The term optimum when used in communication theory is normally interpreted to mean the receiver which minimizes the Bayesian risk for a given set of hypotheses. Unfortunately the selection of the hypothesis is left to the designer so that some variation on optimum designs occur. It is only when we compare the performance of the several optimum systems do we approach the truly optimum system.

The following model is assumed in this paper. The data received at the transmitter is in binary form. The binary data is then converted to one of M code words ($M \geq 2$) to be transmitted. The transmission format is as illustrated below in Figure 1.



where T is the time required to transmit one information bit

z is the transmission pulse width

$M z \leq T \log_2 M$

Figure 1 Pulse Position Modulation

As can be seen in the illustration, we assume that the transmitted pulse of length z is at most equal to $\frac{T\log_2 M}{M}$ so that the transmission symbol rate for M=2 is at most half the data rate. The symbol rate and indeed the duty cycle is seen to decrease as M increases since we assume z to be independent of M. Reiffen and Sherman(1) have shown that performance will improve as the transmission pulse width is decreased and the transmitted energy in the pulse remains constant. Thus, the minimum value of z is determined by the uncertainty principle rather than M. In order to be able to compare the performance of the M'ary systems it is assumed that for all systems the average power is held constant. Then a tradeoff occurs between increased pulse peak power with a decreased average duty cycle factor as the M'ary alphabet increases. Finally we note that in every $T\log_2 M$ seconds one pulse is transmitted. The receiver has to decide which of the preassigned M slots contains the information. Therefore we have a form of pulse position modulation which is roughly the dual of M'ary FSK in more conventional rf systems.

PERFORMANCE

Based on the assumption listed in the introduction, the optimum detector decision criterion and its performance are derived in the appendix. In the optimum detector, every $T\log_2 M$ seconds each of the Mz photon energy measures are compared and the z time slot that contains the greatest number of photons is determined to be the $\log_2 M$ bits of information transmitted in that $T\log_2 M$ time period.*

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*This system compares the energy measure outputs and selects as the signal the one which is greatest. Thus, this detection system is not dependent upon a threshold which is a function of the signal to noise ratio. In the paper by Reiffen and Sherman[1] they showed that the optimum system transmitted only one symbol in each $T\log_2 M$ time interval and in only one z interval. They however implied from this that for the binary case the optimum system transmitted always in the same z interval so that a mark was represented by a pulse and a space by no pulse thus requiring a threshold to determine between mark and space transmissions. In addition, Ross[4] stated that the optimum M'ary system also required a threshold. This need not be the case (M>2) since the system presented in this paper meets the requirements of Reiffen and Sherman's optimality condition (for all M'ary alphabets) without the use of a threshold. This discussion may have its analog in rf systems between the choice of on-off keying or FSK for binary systems and the M'ary extension of FSK (no M'ary extension of on-off keying has seriously been considered).

The derived performance equations in the appendix have been programmed and the results plotted in Figures 2, 3 and 4. In each figure the M'ary symbol error rate is plotted as a function of the average number of signal photons received per bit ($\bar{\alpha}$) for values of M ranging from 2 through 64. Each of the figures differ in that the average number of noise photons received per M'ary symbol $\bar{n}$ is a different value in each figure. It is to be noted that the results are not independent of the absolute values of signal and noise energies as they are in rf communications systems.

In Figures 2 and 3 the results of Ross[2] are plotted for the binary case when on-off pulse transmission is used. The results plotted are for conditions of optimum threshold.* The following conclusions can be obtained after investigation of the graphical results.

1. For a given average noise background level

   a. the keyed binary system with threshold out-performs the no threshold pulse position systems.

   b. An M'ary improvement is obtained as M increases, however, the rate of improvement decreases as M increases in much the same way as MFSK in additive gaussian noise. Thus the most significant gain is obtained in going from M=2 to M=4. It is to be noted that the M=4 system out-performs the on-off keyed binary system.

2. In comparing the system performance as the noise background is varied we see that the rate of improvement at first increases as the noise background increases. That is, in going from M=2 to M=4 we do better on a percentage improvement basis for larger noise backgrounds than for those that are smaller. However, the reverse is true for values of M>4. This is illustrated in Table I.

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*The derivation by Ross is not quite optimum for in the case of a tie at threshold, the selection should be perfectly random. Ross's detector always decides a mark was transmitted in such cases. The difference is at most 1/2 the symbol error rate for a given value of $\bar{\alpha}$.

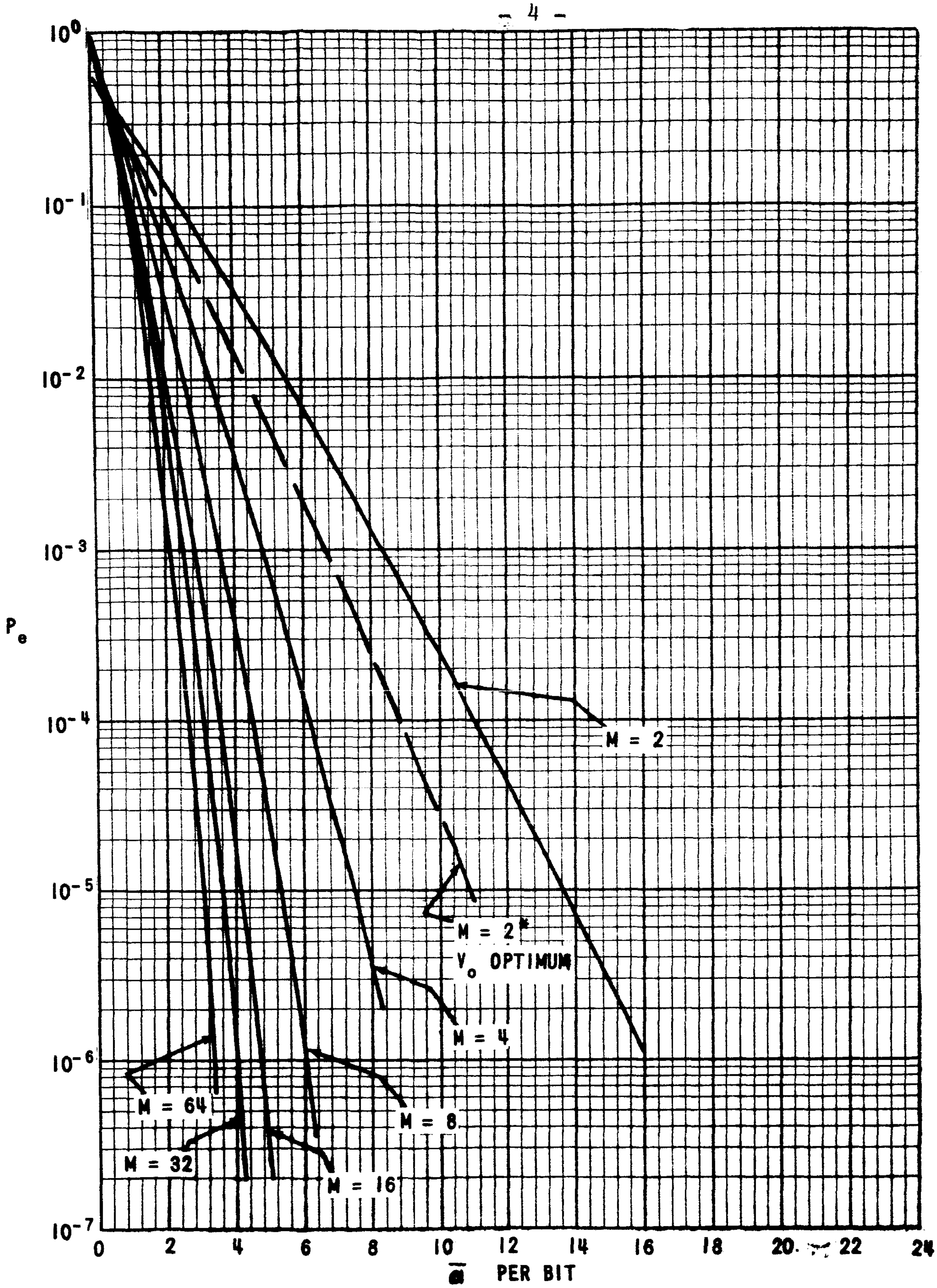

FIGURE 2 - PROBABILITY OF A SYMBOL ERROR P_e AS A FUNCTION OF THE AVERAGE NUMBER OF PHOTONS PER BIT $\overline{m}$ ($\overline{n} = \frac{1}{2}$)

*Curve obtained from Reference 2.

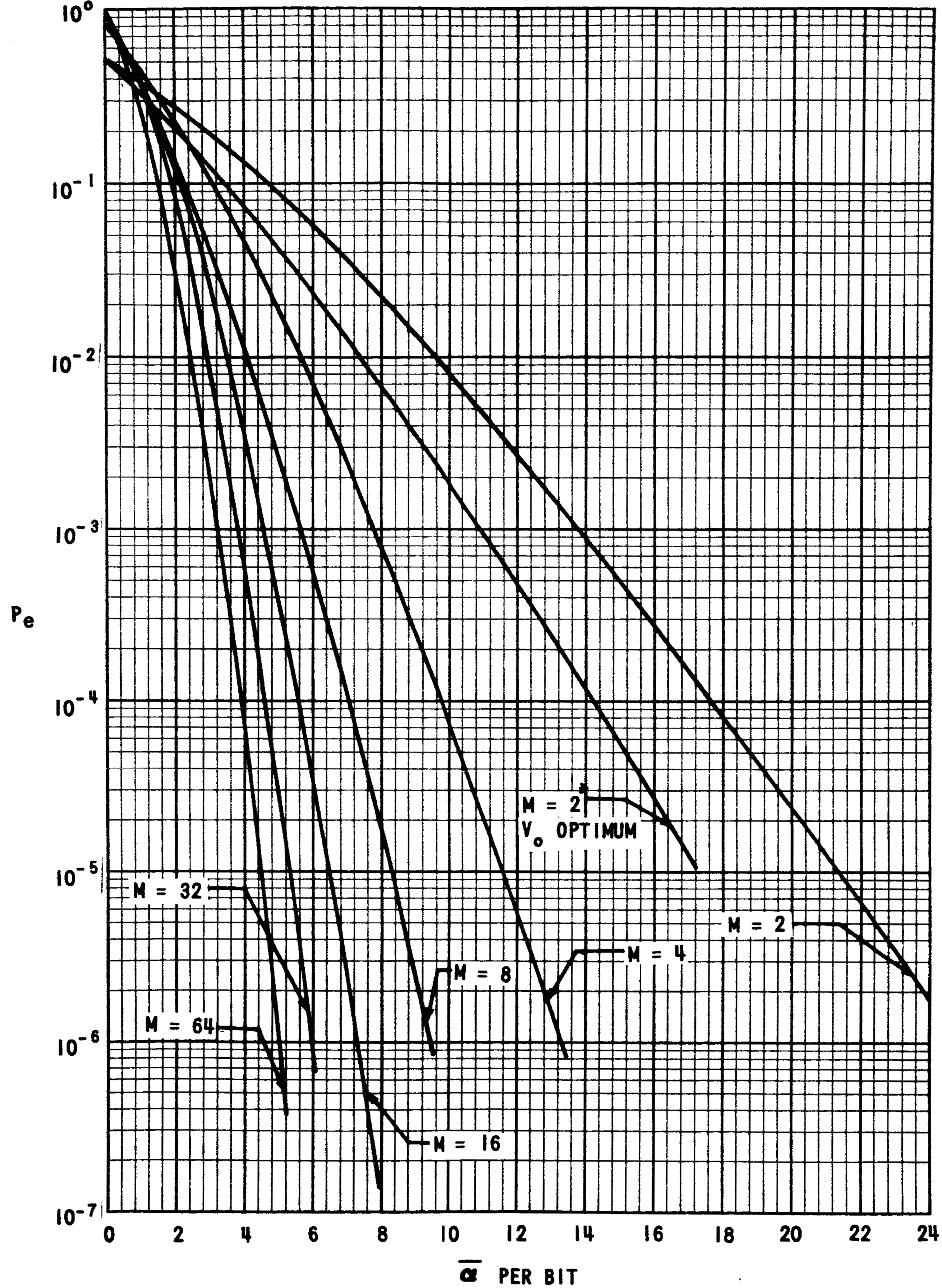

FIGURE 3 - PROBABILITY OF A SYMBOL ERROR P_e AS A FUNCTION OF THE AVERAGE NUMBER OF PHOTONS PER BIT (π = 4)

*Curve obtained from Reference 2.

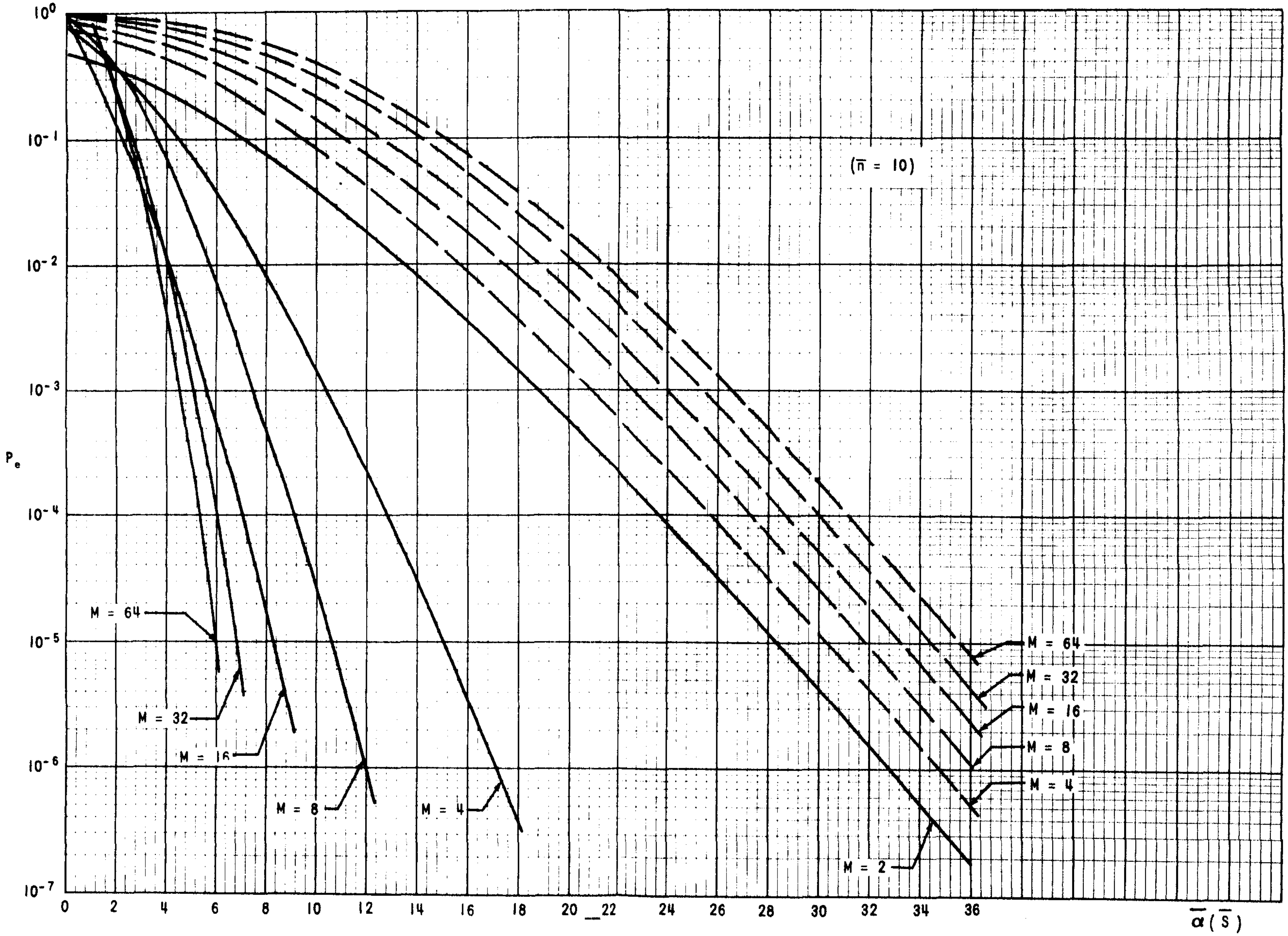

FIGURE 4 - PROBABILITY OF A SYMBOL ERROR P_e AS A FUNCTION OF THE AVERAGE NUMBER OF PHOTONS PER BIT $\overline{\alpha}$ ——— (SYMBOL $\overline{S}$ - - -)

Table 1 M'ary Improvement ($\bar{\alpha}$ - reduction) as a Function of the Normalized Duty Factor for Several Noise Backgrounds

Average Number of Noise Photons	Change in Duty Factor / Duty Factor (M=2) = $\frac{\Delta d}{d}$		
$\bar{n}$	1/2 (M=4)	1/3(M=8)	1/4(M=16)
1/2	5.1 photons 45.4% Reduction	8.3 photons 74% Reduction	9.4 photons 84% Reduction
4	9.8 photons 45.8% Reduction	12.6 photons 55.8% Reduction	14.6 photons 68.2% Reduction
10	13.4 photons 47% Reduction	15.6 photons 55% Reduction	16.8 photons 59. % Reduction

to be expected, the absolute number of average signal photons required for a particular system performance increases as the noise level increases.

L. Schuchman

2034-LS-jr

Attachments
References
Appendix

APPENDIX

DERIVATION OF THE PPM OPTIMUM DETECTOR AND ITS PERFORMANCE

Derivation of the optimum detector [equally likely *a priori* Binary case]

Let us assume our code scheme is as described by Figure 1 with M=2. We further assume that the arrival times of photons for the two z possible signal intervals are Poisson distributed with photon arrivals in the two intervals statistically independent. In addition we assume that the noise and signal processes are independent. The determination of the optimum receiver assuming equally likely transmission of the two symbols, reduces to the computation of the likelihood function $\ell(x)$. This function can be written as

$$\ell(x) = \frac{P[x|\alpha_1(t)]}{P[x|\alpha_2(t)]} > 1 , \qquad \text{(A-1)}$$

where P(x) is the joint distribution for the number of arrivals in the two z intervals.

$\alpha_1(t)$ is the received signal for the transmitted symbol interval (0, T) when the first signal is sent.

$\alpha_2(t)$ is the received signal for the transmitted symbol interval (0, T) when the second signal is sent with P(1) = P(2).

Applying our Poisson assumptions equation (A-1) can be written as

$$\ell(x) = \frac{\dfrac{[\eta(\bar{\alpha}_1+\bar{n})]^{y_1}}{y_1!}\,[e^{-\eta(\bar{\alpha}_1+\bar{n})z}]\,\dfrac{[\eta(\bar{n})]^{y_2}}{y_2!}\,e^{-\eta\bar{n}z}}{\dfrac{[\eta(\bar{\alpha}_2+\bar{n})]^{y_2}}{y_2!}\,e^{-\eta(\bar{\alpha}_2+\bar{n})z}\,\dfrac{[\eta(\bar{n})]^{y_1}}{y_1!}\,e^{-\eta\bar{n}z}} > 1 , \qquad \text{(A-2)}$$

where

$\bar{n}$ is the average number of noise photon arrivals per z interval of time.

$\bar{\alpha}_1$ is the average number of signal photon arrivals per z interval of time when the first signal is transmitted.

$\bar{\alpha}_2$ is the average number of signal photon arrivals per z interval of time when the second signal is transmitted.

η is the quantum efficiency factor of our photon counter device ($\eta \leq 1$).

Assuming that we transmit either signal with the same average energy $\bar{\alpha}_1 = \bar{\alpha}_2 = \bar{\alpha}$, equation (A-2) reduces to

$$\ell(x) = \eta\left(\frac{\bar{\alpha}}{\bar{n}} + 1\right)^{y_1 - y_2} > 1 \qquad \text{(A-3)}$$

Taking logarithms of both sides we have that our decision rule simplifies to

$$y_1 > y_2$$

Thus if y_1 (the actual number of photons received in the z interval of signal one) is greater than y_2 we decide that the first signal was transmitted. If y_2 is greater then we decide the second signal was transmitted. If they are equal, we flip a coin.

M'ary Extension

In the M'ary case we have M, z intervals in which to look for the signal. The natural extension of the binary case for the equally likely transmitted signal case and a "hit or miss" loss matrix is a receiver which decides that signal i is transmitted if

$$y_i = \max(y_1\ y_2\ \cdots\ y_m) \qquad \text{(A-4)}$$

If more than one i satisfies equation (A-4) then we randomly select from these, one as the signal.

To Compute the M'ary Symbol Probability of Error

The probability of error P_e can be written as $1 - P_c$ when P_c is the probability of a correct decision. P_c can be written as (assuming that signal one was transmitted)

$$P_c = P[y_2 < y_1; \; y_3 < y_1; \; \cdots \; y_m < y_1] +$$

$$\frac{1}{2} \binom{M-1}{1} p[y_2 = y_1; \; y_3 < y_1; \; \cdots \; y_m < y_1] +$$

$$\cdots \; \frac{1}{M} \binom{M-1}{M-1} p[y_2 = y_1; \; y_3 = y_1; \; \cdots \; y_m = y_1] \quad , \qquad \text{(A-5)}$$

which can be written more compactly as

$$P_c = \sum_{i=0}^{M-1} \frac{1}{i+1} \binom{M-1}{i} P[y_2 = y_1; \; y_3 = y_1, \; y_{i+1} = y_1, \; y_{i+2} < y_1 \; \cdots \; y_m < y_1]. \qquad \text{(A-6)}$$

Using our independence assumptions we have

$$P_c = \sum_{i=0}^{M-1} \frac{1}{i+1} \binom{M-1}{i} \int_0^{\infty} P(y_1) P^i (y_2 = y_1) P^{M-1-i} [y_2 < y_1] dy_1 \qquad \text{(A-7)}$$

Substituting into equation (A-7) the Poisson distribution for the several y_i results in:

$$P_c = \sum_{i=0}^{M-1} \frac{1}{i+1} \binom{M-1}{i} \sum_{y_1=\varepsilon_i}^{\infty} [\exp-(\bar{n}+\bar{\alpha})] \frac{[\bar{n}+\bar{\alpha}]^{y_1}}{y_1!} \left[\frac{(\exp-\bar{n})\ \bar{n}^{y_1}}{y_1!}\right]^i \cdot$$

$$\left[\sum_{y_2=0}^{y_1-\varepsilon_i} \frac{[\exp-\bar{n}]\ \bar{n}^{y_2}}{y_2!}\right]^{M-1-i} \qquad (A-8)^*$$

where $\varepsilon_i = \begin{cases} 0 & i=M-1 \\ 1 & i \neq m-1 \end{cases}$

---

*It is assumed the photon efficiency factor η is reflected in the parameters $\bar{n}$ and $\bar{\alpha}$.

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