TRANSLATION

REPRESENTATION OF THE TRANSFER FUNCTIONS OF GIVEN STRUCTURES BY THEIR EIGEN-VIBRATIONS, TAKING INTO CONSIDERATION ANY (ARBITRARY) EXCITATION

Jurgen Wolfgang Schindelin

Translation of "Darstellung der Ubertragungs-funktcionen vorgegebner Konstruktionen durch deren Eigen-schwingungen mit Berucksichtigung belieber Erregungen"
VDI Zeitschrift, Vol. 110, No. 8, 1968, pp 307-310

Translated for Bellcomm, Inc, Washington, D.C. 20036
NASw-417

From: VDI-Z 1968, 110 (8):307-310.

TITLE

Representation of the transfer functions of given structures by their eigen-vibrations, taking into consideration any (arbitrary) excitations.

By Jürgen Wolfgang Schindelin, Washington, D. C.

The dynamic behavior of given designs or structures can be described by a system of differential equations. The solution obtained consists in sums of integral terms that can be developed for the eigen-vibrations of the structure. In that way, it becomes possible to derive the transfer functions of the structural elements and to examine their dynamic behavior as dependent on any (arbitrary) excitation functions, by means of the methods commonly used in control technology.

Frequently, it is very important to know the dynamic transfer behavior of given designs or structures, due to the effect of the excitations affecting them. And that is, indeed, of importance when, e.g., the following has to be investigated:

1. The chronological behavior of the elastic deformations of a structure as well as the stability stresses caused by them, as dependent on the course of the excitations;

2. the effect of those elastic deformations on the output signals of sensors that are located within the structure in question (such sensors are gyroscopes and acceleration pick-ups in a flying body [1;2]);

3. the laws of motion of such a structure which serves, itself, as sensor (such sensors are antennae that localize moving targets in space [3]);

4. the behavior of structures within the time range or frequency range as dependent on limiting quantities (such structures are the transfer elements in control systems).

The movements of the elements of a given structure due to the effect of any (arbitrary) excitation functions can be described by a system of differential equations. The solution obtained consists in sums of integral terms that can be developed for the eigen-vibrations of the structure. In that way, it becomes possible, by the use of the Laplace transformation, to derive the transfer functions of the structural elements.

Dr. ing. Jürgen Wolfgang Schindelin, M.S., was a staff member of the Central Institute for Research and Development of the firm of Fried. Krupp at Essen.

By means of the transfer functions, it is possible to investigate the dynamic behavior of the structure as affected by any (arbitrary) excitation functions, by the use of the methods applied commonly in control technology. The excitations may be present as determined functions of time, or else as chance functions, e.g., as squall spectra (cf., e.g., [4,5]).

Derivation of the equations of motion

In the derivation of the equations of motion, we first replace the given structure by discrete masses the number, distribution, and amounts of which will have to be chosen appropriately (cf. [6,7].

When i is an index that is able to assume the values of i = 1, 2, ..., n, then the displacements X_i of the n discrete masses m_i as well as the excitations P_i affecting them are connected with one another by the equation

$$\mathfrak{X} = \mathfrak{C}\mathfrak{P} \tag{1}$$

(cf. [7, 8]). In that equation, X denotes the column matrix associated with the displacements X_i, viz.

$$\mathfrak{X} = \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix} \tag{1a}$$

P denotes the column matric associated with the excitations P_i, viz.

$$\mathfrak{P} = \begin{pmatrix} P_1 \\ \vdots \\ P_n \end{pmatrix} \tag{1b}$$

and C denotes the square matrix, viz.

$$\mathfrak{C} = \begin{pmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & & \vdots \\ c_{n1} & \cdots & c_{nn} \end{pmatrix} \tag{1c}$$

the elements c_{il} of which - with l = 1, 2, ..., n - are the limiting numbers of the displacements that are caused by a "standardized" excitation in the point i that attacks in the point l. By multiplication by the matrix that is inverse to the matrix C, viz.

$$\mathfrak{C}^{-1} = \begin{pmatrix} c_{11}^{(-1)} & \cdots & c_{1n}^{(-1)} \\ \vdots & & \vdots \\ c_{n1}^{(-1)} & \cdots & c_{nn}^{(-1)} \end{pmatrix} \tag{1d}$$

we shall obtain the equation

$$\mathfrak{P} = \mathfrak{C}^{-1}\mathfrak{X} \tag{2}$$

from the Equation (1); it describes the static behavior of the system in question.

The dynamic behavior may be understood by a system of n differential equations. To that end, we shall find that

$$\mathfrak{P}(t) = \left[\frac{d^2}{dt^2}\mathfrak{M} + \mathfrak{C}^{-1}\right]\mathfrak{X}(t) \tag{3}$$

in that equation, d^2/dt^2 denotes a differential operator; M denotes the diagonal matrix, viz.

$$\mathfrak{M} = \begin{pmatrix} m_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & m_n \end{pmatrix} \tag{4}$$

and X(t) and P(t) denote the column matrix according to Equations (1a) and (1b), respectively, the elements $X_i \equiv X_i(t)$ and $P_i \equiv P_i(t)$, respectively, now depend on the time t.

It may be mentioned that, on the basis of a once chosen spatial arrangement of the discrete masses m_i and of the kinematic conditions resulting from it for their motions, frequently not only diagonal elements, but also non-diagonal elements may appear in the matrix M. Since they also can be multiplied, in all cases, by the operator d^2/dt^2, the method developed can be applied in this case also.

By the use of the Laplace transformations $L\{P_i(t)\} \equiv p_i(s)$ and $L\{X_i(t)\} \equiv x_i(s)$ which associate the variables $p_i(s)$ and $x_i(s)$ in the picture range or frequency range with the variables $P_i(t)$ and $X_i(t)$ in the chronological sphere, it follows from the Equation (3) that there will have to be

$$\mathfrak{p}(s) = [s^2 \mathfrak{M} + \mathfrak{C}^{-1}]\, \mathfrak{x}(s) \tag{5}$$

when

$$\mathfrak{x}(s) = \begin{pmatrix} x_1(s) \\ \vdots \\ x_n(s) \end{pmatrix} \tag{5a}$$

and

$$\mathfrak{p}(s) = \begin{pmatrix} p_1(s) \\ \vdots \\ p_n(s) \end{pmatrix} \tag{5b}$$

and when s = d/dt is the operator of the Laplace-transformation. By solving the system of equations that corresponds to the Equation (5), we shall obtain the n displacements, viz.

$$x_i(s) = \sum_{i=1}^{n} \frac{G_{i,i}(s)}{N(s)}\, p_i(s) \tag{6}$$

within the picture range, on the basis of Cramer's Rule. In that Equation,

$$N(s) = \det(s^2 \mathfrak{M} + \mathfrak{C}^{-1}) \tag{6a}$$

denotes the determinant of the system, and

$$G_{i,i}(s) = (-1)^{i+i} D_{i+i}(s) \tag{6b}$$

will apply when D_{i+i} is the determinant of the (n — 1)-th order, that is derived from N (s) by striking out the i-th line and the i-th column. The formation of sums in the Equation (6) extends only over the first index (line index) while the second index (column index) remains constant for a specified i. For that reason, we shall now simply write $G_{i,i}$ instead of G_i, so as to simplify the notation. Of course, we assume on the basis of our premise that N (s) $\neq$ 0.

The determinant N (s) may be developed into a polynomial having the general form of

$$N(s) = a_{2n} s^{2n} + a_{2n-2} s^{2n-2} + \ldots + a_2 s^2 + a_0 \quad , \tag{7}$$

when a_0, a_2, ..., a_{2n} is the real coefficient of development. The polynomial has exactly 2 n roots of s_v, with the index of v = 1, 2, ..., 2 n. In the present case, the roots are, in pairs, conjugate-complex, which means that there are n roots having the form of

$$s_\mu = j\omega_\mu \tag{8a}$$

and n roots having the form of

$$s_{\mu+n} = -j\omega_\mu \, . \tag{8b}$$

when the index μ is equal to 1, 2, ..., n, and when j is the imaginary unit. The polynomial may be reduced to the product form of

$$N(s) = a_{2n}(s - s_1)\ldots(s - s_n) \times \\ \times (s - s_{n+1})\ldots(s - s_{2n}) \quad . \tag{8c}$$

<u>Development for eigen-vibrations</u>

First, it is possible, by splitting it up into partial fractions, to reduce the quotient G_i (s)/N (s) that enters into the summation, in accordance with the Equation (6), to the form of

$$\frac{G_i(s)}{N(s)} = \frac{A_{i,1}}{s - s_1} + \ldots + \frac{A_{i,n}}{s - s_n} + \\ + \frac{A_{i,n+1}}{s - s_{n+1}} + \ldots \frac{A_{i,2n}}{s - s_{2n}} \tag{9}$$

when

$$A_{i,\mu} = \frac{G_i(s_\mu)}{N(s_\mu)}(s - s_\mu) \tag{10}$$

and

$$A_{i,\mu+n} = \frac{G_i(s_{\mu+n})}{N(s_{\mu+n})}(s - s_{\mu+n}). \tag{11}$$

represent a total of 2n coefficients. These coefficients are, in pairs, conjugate-complex, corresponding to

$$A_{i,\mu} = \alpha_{i\mu} + j\beta_{i\mu} \, . \tag{12}$$

and

$$A_{i,\mu+n} = \alpha_{i\mu} - j\beta_{i\mu} \tag{13}$$

Since the roots s_v are also conjugate-complex, in pairs, according to the Equation (8a) and (8b), we shall find that

$$\mathfrak{L}^{-1}\left(\frac{A_{i,\mu}}{s-s_\mu}+\frac{A_{i,\mu+n}}{s-s_{\mu+n}}\right)= = 2(\alpha_{i\mu}\cos\omega_\mu t-\beta_{i\mu}\sin\omega_\mu t) \tag{14}$$

when L^{-1} is the re-transformation into the chronological sphere (see [8]). There will be:

$$2(\alpha_{i\mu}\cos\omega_\mu t-\beta_{i\mu}\sin\omega_\mu t)= = B_{i\mu}\sin(\omega_\mu t+\Gamma_{i\mu}) \tag{15}$$

when

$$B_{i\mu}=2\sqrt{\alpha_{i\mu}^2+\beta_{i\mu}^2}\ .\ . \tag{15a}$$

and

$$\Gamma_{i\mu}=\operatorname{arc\,tan}\left(-\frac{\alpha_{i\mu}}{\beta_{i\mu}}\right) \tag{15b}$$

We shall, therefore, find that

$$\mathfrak{L}^{-1}\left(\frac{G_i(s)}{N(s)}\right)=\sum_{\mu=1}^{n}B_{i\mu}\sin(\omega_\mu t+\Gamma_{i\mu}) \tag{16}$$

The term

$$\mathfrak{L}^{-1}\left(\frac{G_i(s)}{N(s)}p_i(s)\right)=\sum_{\mu=1}^{n}B_{i\mu}\sin(\omega_\mu t+\Gamma_{i\mu})*P_i \tag{17}$$

follows from the Equations (6) and (16) and pertains to the effect of the excitation P_i on the chronological course of the deflection X_i as developed for the eigen-vibrations of the given structures. In (17), the symbol * denotes a convolution integral (cf. [8]).

The total deflection of the mass m_i may be obtained by summation, i.e., we shall find that

$$X_i(t)=\sum_{i=1}^{n}\sum_{\mu=1}^{n}B_{i\mu}\sin(\omega_\mu t+\Gamma_{i\mu})*P_i \tag{18}$$

This Equation applies to the undamped system.

The structural damping which has been neglected up to this point, can be taken into consideration by means of the subsequent introduction of the coefficient $K_\mu \geqslant 0$. Then we shall find that

$$X_i(t)=\sum_{i=1}^{n}\sum_{\mu=1}^{n}B_{i\mu}e^{-K_\mu t}\sin(\omega_\mu t+\Gamma_{i\mu})*P_i \tag{19}$$

When the excitations are, in particular, impulse functions according to the equation

$$P_i(t)=Q_i\delta(t)$$

in which Q_i are n constant values and $\delta(t)$ is Dirac's δ-function, then we shall have

$$X_i(t)=\sum_{i=1}^{n}\sum_{\mu=1}^{n}Q_i B_{i\mu}e^{-K_\mu t}\sin(\omega_\mu t+\Gamma_{i\mu}) \tag{20}$$

for the deflections within the chronological sphere. That follows directly from the Equation (19).

Determination of the transfer functions

By means of the Equation (19), we shall now determine the general transfer function for the behavior of the deflections X_i in the case of any (arbitrary) excitations P_i.

Since (cf. [8]) the Laplace-transformed equation is

$$\mathfrak{L}\{e^{-at}F(t)\} = f(s+a),$$

it follows that

$$\mathfrak{L}\{e^{-K_\mu t}\sin(\omega_\mu t + \Gamma_{i\mu})\} = \frac{\omega_\mu \cos\Gamma_{i\mu} + (s+K_\mu)\sin\Gamma_{i\mu}}{(s+K_\mu)^2 + \omega_\mu^2} \tag{21}$$

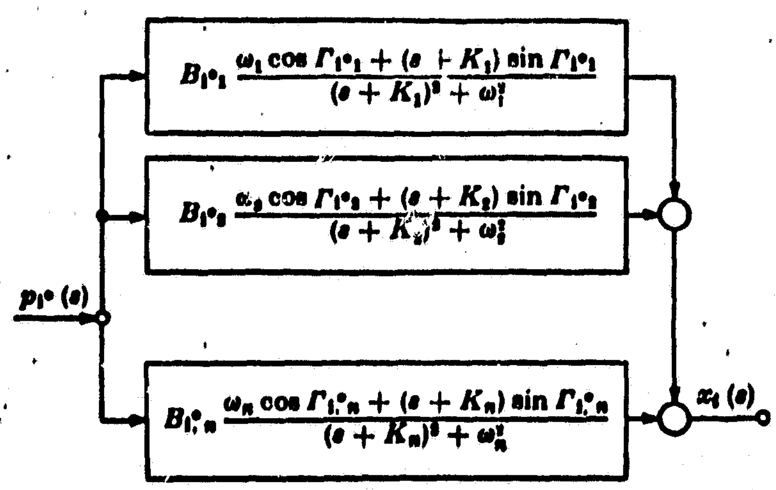
will have to be valid. Hence, however, the Equation

$$x_i(s) = \sum_{i=1}^{n}\sum_{\mu=1}^{n} B_{i\mu}\frac{\omega_\mu \cos\Gamma_{i\mu} + (s+K_\mu)\sin\Gamma_{i\mu}}{(s+K_\mu)^2 + \omega_\mu^2}\,p_i(s) \tag{22}$$

follows from the Equation (19), for the general transfer function.

The Signal flow chart 1 indicates the transfer function for the case when only the excitation P_{i*} affects the structure, i.e., when $P_i = 0$ applies to all $i = 1^*$. This transfer function will be as follows, according to the Equation (22):

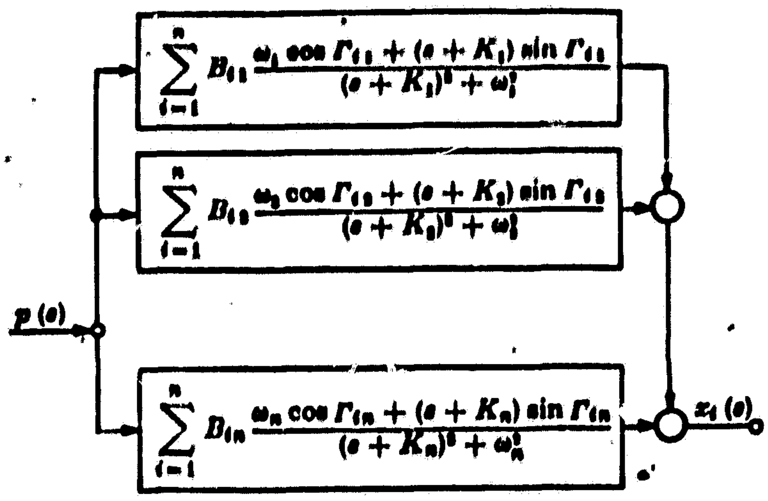
$$x_i(s) = \sum_{\mu=1}^{n} B_{i^*\mu}\frac{\omega_\mu \cos\Gamma_{i^*\mu} + (s+K_\mu)\sin\Gamma_{i^*\mu}}{(s+K_\mu)^2 + \omega_\mu^2}\,p_{i^*}(s) \tag{23}$$



Signal flow chart 1. Transfer function of $x_i(s)$ according to the Equation (23), for the case when only the excitation $p_{i*}(s)$ affects the structure.

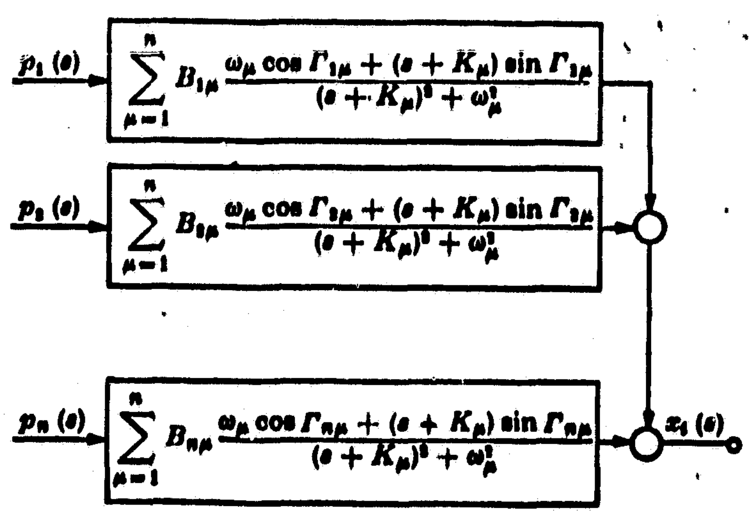
In the case of the transfer function as shown in the Signal flow chart 2, it was assumed that all excitation functions are equal, i.e., that $P_1 = P_2 = \dots = P_n = P$ and that, therefore,

$$x_i(s) = \sum_{i=1}^{n} \sum_{\mu=1}^{n} B_{i\mu} \frac{\omega_\mu \cos \Gamma_{i\mu} + (s + K_\mu) \sin \Gamma_{i\mu}}{(s + K_\mu)^2 + \omega_\mu^2} \, p(s) \tag{24}$$



Signal flow chart 2. Transfer function of x_i (s) according to the Equation (24) for the case when the excitations p_i (s) ≡ p (s) affect the structure for all i.

will apply also. Finally, the Signal flow chart 3 shows the complete transfer function according to the Equation (22) for any (arbitrary) excitation functions.



Signal flow chart 3. Transfer function of x_i(s) according to the Equation (22), for the case when any (arbitrary) excitations affect the structure.

Taking the initial conditions into account.

Now, the deflections X_i(t) of the point masses of m_i and the lateral deviations of X_i(t) at the moment of t = 0, i.e., the initial conditions of

$X_i(0)$ and of $\dot{X}_i(0)$, will be taken into account. The Laplace-transformed equation of the second chronological derivation of $X_i(t)$ is determined by

$$\mathfrak{L}\left(\frac{d^2 X_i}{dt^2}\right) = s^2 x_i(s) - s X_i(0) - \dot{X}_i(0)$$

(cf. [8]). Therefore, the Laplace-transformed equation will be

$$\mathfrak{L}\left(\frac{d^2}{dt^2}\mathfrak{M}\mathfrak{X}(t)\right) = s^2 \mathfrak{M} \mathfrak{x}(s) - s\mathfrak{G} - \mathfrak{H} \qquad (25)$$

In that Equation, M, X(t), and x(s) denote the matrices which have already been explained, and

$$\mathfrak{G} = \begin{pmatrix} m_1 X_1(0) \\ \vdots \\ m_n X_n(0) \end{pmatrix} \qquad (25a)$$

and

$$\mathfrak{H} = \begin{pmatrix} m_1 \dot{X}_1(0) \\ \vdots \\ m_n \dot{X}_n(0) \end{pmatrix} \qquad (25b)$$

denote column matrices. Using these expressions, we shall now obtain, in lieu of the Equation 5, the following Equation within the picture range:

$$\mathfrak{p}(s) + s\mathfrak{G} + \mathfrak{H} = [s^2 \mathfrak{M} + \mathfrak{C}^{-1}]\,\mathfrak{x}(s) \qquad (26)$$

That Equation can again be solved by the application of Cramer's Rule, for the deflections of $x_i(s)$. The expressions corresponding to the Equation (6) will then contain the initial values of $X_i(0)$ and $\dot{X}_i(0)$ and will make it possible to compute the chronological course of the deflections in the case of any arbitrarily given initial values. Since in the present case, too, the polynomial N(s) as explained by the Equation (6) appears, within the relations that correspond to the Equation (6), in the denominator, the structure possesses - as is required - the same eigen-frequencies.

Literature

(Books are identified by •)

[1] • *Schindelin, J.*: Zur Systemanalyse ferngelenkter stabilisierter Flugkörper. Jahrbuch der WGLR 1963. Braunschweig: Friedr. Vieweg und Sohn, 1964.

[2] • *Koelle, H. H.*: Handbook of Astronautical Engineering. New York: McGraw Hill 1961.

[3] *Schindelin, J.*: Regelungsaufgaben in der Flugkörpertechnik. Industrieanzeiger, Antrieb–Getriebe–Steuerung 88 (1966) Nr. 23, S. 1898/1902.

[4] • *Truxal, J.*: Automatic Feedback Control System Synthesis. New York: McGraw Hill 1955.

[5] • *Solodownikow, W. W.*: Grundlagen der selbsttätigen Regelung. Bd. I. München: R. Oldenbourg, und Berlin: VEB Verlag Technik 1959.

[6] • *Hohenemser, K.*, u. *W. Prager*: Dynamik der Stabwerke. Berlin: Springer Verlag 1933.

[7] • *Öry, H., E. Hornung* u. *H. Ludwig*: Berechnung der Eigenschwingungsformen und Frequenzen bei Flugkörpern als Balken und als Schale mit beliebigen Randbedingungen. Jahrbuch der WGLR 1964, Braunschweig: Friedr. Vieweg und Sohn 1965.

[8] • *Doetsch, G.*: Anleitung zum praktischen Gebrauch der Laplace-Transformation. München: R. Oldenbourg 1956.

A 19485