BELLCOMM, INC.

SUBJECT: A Note on a Solution of the Three Body Problem Case- 218

DATE: December 13, 1965

FROM: R. Y. Pei

ABSTRACT

26251

The Equilateral Triangle Solution of the three body problem is derived in a simple manner. The geometric properties inherent in such a configuration are discussed. The problem is pertinent to earth-moon libration point missions that have been suggested for the Apollo Applications Program.

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MEMORANDUM FOR FILE

INTRODUCTION

The Equilateral Triangle Solutions of the finite three body problem have been traditionally based upon the work of Lagrange in his prize memoir, "Essai sur le Problème des Trois Corps".[1] His method required the derivation of three differential equations involving the three mutual distances. One of these equations was of the third order, and the remaining two of the second order each, making the whole problem of the seventh order. The differential equations were then integrated by assuming that the ratios of the mutual distances were constants. The present note attempts to derive the same solution in a simpler manner and bring out the geometric properties inherent in such a configuration. The problem is of interest in view of some earth-moon libration point missions that have been suggested for Apollo Applications.

PROOF

Assume that there exists a stable plane configuration in which three finite bodies revolve about a given point.

In the absence of any disturbances, the centrifugal forces are counterbalanced by the mutual attractions among the three bodies, such mutual attractions being equal and opposite for each pair. Figure 1 depicts such a system.

Lemma- If the forces of attraction between three bodies obey Newton's third law, then the three resultant forces are concurrent, regardless of the geometry of the triangle formed by the three bodies as vertices. The proof is given in Appendix I.

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[1] Coll. Works, Vol. VI., p. 229. Tisserand's Mec. Cel. Vol. I., Chap. VIII.

In order to derive the equilateral triangle solution, the origin, 0, is taken as the barycenter.[1] By definition, therefore, the vector sum of the three quantities $M_1\vec{r}$, $M_2\vec{s}$ and $M_3\vec{\chi}$ must form a closed triangle, as shown in Figure 2.

Application of the Sine Law now yields the following:

$$a : b : c = \sin\alpha : \sin\beta : \sin(\alpha+\beta) \tag{1}$$

$$r : s : c = \sin(\beta-m) : \sin\ell : \sin(\ell+\beta-m) \tag{2}$$

$$r : b : \chi = \sin\theta : \sin p : \sin(\alpha-\ell) \tag{3}$$

$$s : a : \chi = \sin(\alpha+\beta+\theta) : \sin q : \sin m \tag{4}$$

$$M_1 r : M_2 s : M_3 \chi = \sin q : \sin p : \sin(\ell+\beta-m) \tag{5}$$

$$|\vec{f}_1| : |\vec{f}_2| : |\vec{f}_3| = [\sin(\alpha-\ell)\sin(\beta-m)] : [\sin\ell \sin m] : [\sin m \sin(\alpha-\ell)] \tag{6}$$

The magnitude of the attractive forces, according to Newton's law, are also related in the following manner:

$$|\vec{f}_1| : |\vec{f}_2| : |\vec{f}_3| = \frac{M_2 M_3}{a^2} : \frac{M_1 M_3}{b^2} : \frac{M_1 M_2}{c^2} \tag{7}$$

Equating (6) and (7), the following expression is obtained for mass ratio $\frac{M_2}{M_3}$:

$$\frac{M_2}{M_3} = \frac{c^2}{b^2} \frac{\sin(\alpha-\ell)}{\sin\ell} \tag{8}$$

From Equation (5), the same mass ratio is obtained as follows:

$$\frac{M_2}{M_3} = \frac{\chi}{s} \cdot \frac{\sin p}{\sin(\ell+\beta-m)} \tag{9}$$

---

(1) The same reference point is used in all classical proofs and follows immediately from the absence of external forces.

From Equations (2) and (3),

$$\frac{s}{c} = \frac{\sin \ell}{\sin (\ell+\beta-m)}$$

$$\frac{b}{x} = \frac{\sin p}{\sin (\alpha-\ell)}$$

or,

$$\frac{x}{s} = \frac{b}{c} \; \frac{\sin (\ell+\beta-m) \sin (\alpha-\ell)}{\sin p \sin \ell} \qquad (10)$$

Substituting (10) into (9) and equating with (8), the following is obtained:

$$\frac{M_2}{M_3} = \frac{c^2}{b^2} \; \frac{\sin (\alpha-\ell)}{\sin \ell}$$

$$= \frac{b}{c} \; \frac{\sin (\alpha-\ell)}{\sin \ell}$$

or,

$$b = c \qquad (11)$$

Similarly, from Equations (6) and (7),

$$\frac{M_1}{M_2} = \frac{b^2}{a^2} \cdot \frac{\sin \ell \sin m}{\sin (\alpha-\ell) \sin (\beta-m)} \qquad (12)$$

Equation (5) yields,

$$\frac{M_1}{M_2} = \frac{s}{r} \; \frac{\sin q}{\sin p} \qquad (13)$$

Equations (2), (3) and (4) yield,

$$\frac{s}{r} = \frac{a}{b} \; \frac{\sin (\alpha+\beta+\theta)}{\sin \theta} \; \frac{\sin p}{\sin q}$$

$$= \frac{a}{b} \; \frac{\sin \ell \sin m}{\sin (\alpha-\ell) \sin (\beta-m)} \; \frac{\sin p}{\sin q} \qquad (14)$$

Substituting (14) into (13) and equating with (12), the following is obtained:

$$\frac{M_1}{M_2} = \frac{b^2}{a^2} \; \frac{\sin \ell \sin m}{\sin (\alpha-\ell) \sin (\beta-m)}$$

$$= \frac{a}{b} \; \frac{\sin \ell \sin m}{\sin (\alpha-\ell) \sin (\beta-m)}$$

or,

$$a = b \tag{15}$$

Equalities (11) and (15) prove the particular solution in question.

Dynamic Equilibrium

The equilateral triangle solution is sufficient to insure dynamic equilibrium provided the system is given a suitable angular velocity of rotation. This can also be shown from the geometric relationships derived in this note. Specifically, the following proof will be given.

Let Ω_1, Ω_2 and Ω_3 be three angular velocities of rotation necessary to achieve dynamic equilibrium of the three masses under consideration. It will be shown that these three angular velocities are equal.

In order to achieve dynamic equilibrium for M_1, the following condition has to be met:

$$M_1 \Omega_1^2 r = |\vec{f}_2 + \vec{f}_3| = |\vec{f}_2| \; \frac{\sin \alpha}{\sin \ell} \tag{16}$$

Similarly, for M_2 and M_3, we have

$$M_2 \; \Omega_2^{\,2} \; s = |\vec{f}_3 + \vec{f}_1| = |\vec{f}_3| \; \frac{\sin \beta}{\sin m} \tag{17}$$

$$M_3 \; \Omega_3^{\,2} \; x = |\vec{f}_1 + \vec{f}_2| = |\vec{f}_1| \; \frac{\sin \gamma}{\sin \theta} \tag{18}$$

From equations (16) and (17), we obtain

$$\frac{\Omega_1^{\,2}}{\Omega_2^{\,2}} = \frac{|\vec{f}_2|}{|\vec{f}_3|} \; \frac{M_2 s}{M_1 r} \; \frac{\sin \alpha}{\sin \beta} \; \frac{\sin m}{\sin \ell} \tag{19}$$

From equations (5) and observing that $\alpha=\beta$, we obtain

$$\frac{\Omega_1^{\,2}}{\Omega_2^{\,2}} = \frac{|\vec{f}_2|}{|\vec{f}_3|} \; \frac{\sin p}{\sin q} \; \frac{\sin m}{\sin \ell} \tag{20}$$

Equations (3), (4), (6) and (15) yield the following:

$$\frac{\sin p}{\sin q} = \frac{\sin (\alpha-\ell)}{\sin m} \tag{21}$$

$$\frac{|\vec{f}_2|}{|\vec{f}_3|} = \frac{\sin \ell}{\sin (\alpha-\ell)} \tag{22}$$

Substituting (21) and (22) into equation (20), we obtain:

$$\Omega_1 = \Omega_2 \tag{23}$$

Similarly, with equations (16) and (18), we may write,

$$\frac{\Omega_1^{\,2}}{\Omega_3^{\,2}} = \frac{|\vec{f}_2|}{|\vec{f}_1|} \; \frac{M_3 \chi}{M_1 r} \; \frac{\sin \theta}{\sin \ell} \tag{24}$$

Noting that a = b = c, we obtain from equation (7)

$$\frac{M_3}{M_1} = \frac{|\vec{f}_1|}{|\vec{f}_3|} \tag{25}$$

From equation (3):

$$\frac{x}{r} = \frac{\sin (\alpha - \ell)}{\sin \theta} \tag{26}$$

Substituting equations (25) and (26) into (24), and noting equation (22), we obtain:

$$\Omega_1^2 = \Omega_3^2 \tag{27}$$

1013-RYP-taa

R. Y. Pei

Attachment
Appendix I

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APPENDIX I

In this Appendix it is proven that the dynamic equilibrium of three bodies revolving about a given point in space is indifferent to the shape of the triangle formed by the three bodies as vertices, as long as the mutual attractions between bodies are equal and opposite. The problem is restated as follows:

Let an arbitrary point 0 be chosen within the triangle (Figure 1). Let equal segments $|\vec{f}_3|$ be marked off from the two base vertices as shown. Subtract $\vec{f}_3$ and $-\vec{f}_3$ from resultants which must lie along $M_1 0$ and $M_2 0$ respectively to assure dynamic equilibrium, to yield $\vec{f}_2$ and $\vec{f}_1$ that are in the directions of $M_1 M_3$ and $M_2 M_3$. The theorem will be proven if it can be shown that the angle θ formed by $M_1 M_3$ and $0M_3$ is such that if a segment equal to $|\vec{f}_2|$ is marked off from M_3, then its subtraction from a resultant that lies along $M_3 0$ will yield a force equal and opposite to $\vec{f}_1$ in the direction of $M_2 M_3$.

PROOF: Applying the Sine law to triangles $M_1 0 M_3$, $M_2 0 M_3$ and $M_1 0 M_2$, the following is obtained:

$$\frac{r}{\sin \theta} = \frac{\chi}{\sin (\alpha - \ell)}$$

$$\frac{s}{\sin (\gamma - \theta)} = \frac{\chi}{\sin m}$$

$$\frac{r}{\sin (\beta - m)} = \frac{s}{\sin \ell}$$

or,

$$\frac{\sin \theta}{\sin (\gamma - \theta)} = \frac{\sin (\alpha - \ell) \sin (\beta - m)}{\sin \ell \sin m} \tag{A-1}$$

From the force diagrams at the base of the triangle, the following may be written:

$$\frac{|\vec{f}_2|}{\sin \ell} = \frac{|\vec{f}_3|}{\sin (\alpha-\ell)}$$

$$\frac{|\vec{f}_1|}{\sin (\beta-m)} = \frac{|\vec{f}_3|}{\sin m}$$

or,

$$\frac{|\vec{f}_1|}{|\vec{f}_2|} = \frac{\sin (\alpha-\ell) \sin (\beta-m)}{\sin \ell \sin m} \qquad \text{(A-2)}$$

Comparing Equations (A-1) and (A-2), it follows that

$$\frac{|\vec{f}_1|}{|\vec{f}_2|} = \frac{\sin \theta}{\sin (\gamma-\theta)}$$

which proves the theorem.

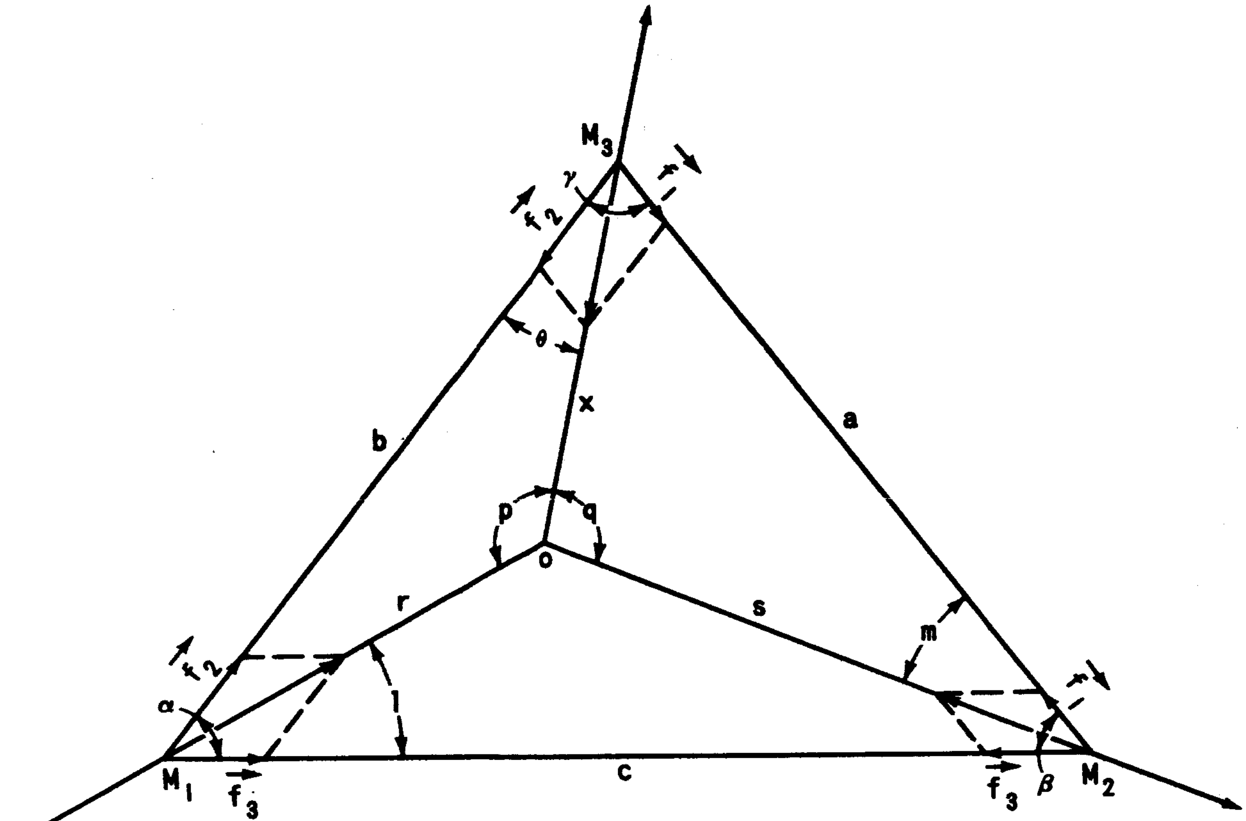

FIGURE I FORCE DIAGRAM

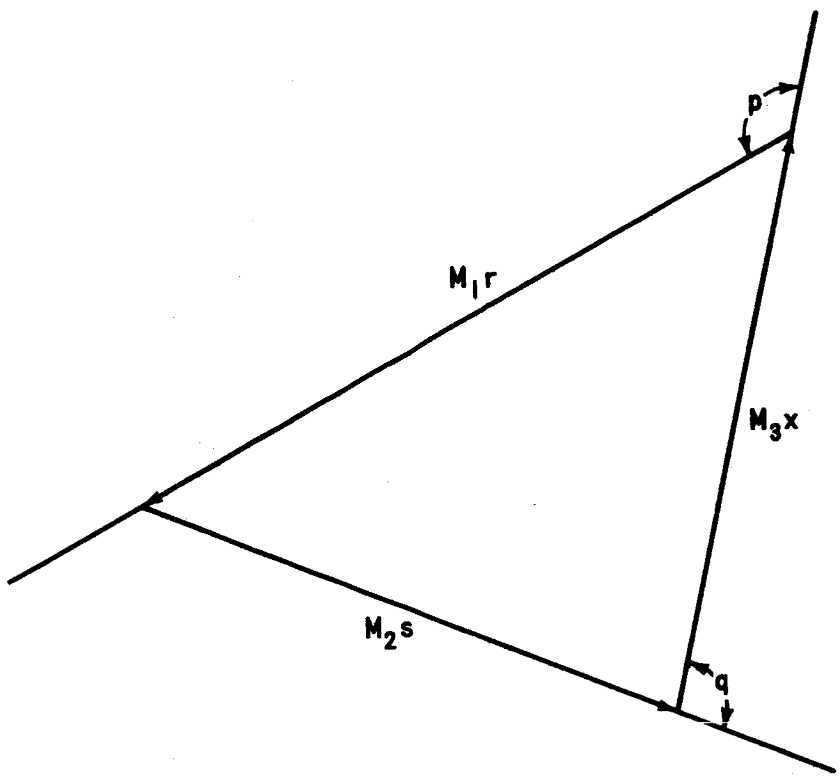

FIGURE 2 MOMENT VECTOR DIAGRAM