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ON THE OPTIMUM STABILIZATION OF SOLID BODY'S ROTARY MOTION WITH THE AID OF A GUIDED GYROSCOPE

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SUMMARY

The solution of the problem of optimum stabilization of the equilibrium position of a free solid body with the help of a guided gyroscope installed on the body, is given in the work reference [1].

It is shown below that, with the aid of the very same means, one may achieve the optimum, in the sense of minimum of a certain functional, stabilization of the stationary motion of a solid body, constituting a permanent rotation around the axis of dynamic symmetry.

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1. Statement of the Problem

Let us consider a free solid body, whose main central axes of inertia are directed along the axes of coordinates $Ox_1x_2x_3$. A balanced gyroscope is so installed on the body on a Cardan (universal) joint, that the axis of the external ring is directed along the straight line Ox_1, while the fixed point coincides with the mass center O of the body. The gyroscope is guided with the aid of three motors, inducing moments with respect to the axes of the external and internal rings and to rotor axis.

Following were the transformed equations of motion of the examined mechanical system, considered in the work [1], without taking into account the mass of the Cardan suspension:

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$$\begin{aligned} A_1 p_1^{\bullet} &= (A_2 - A_3) p_2 p_3 - u_1 \\ A_2 p_2^{\bullet} &= (A_3 - A_1) p_3 p_1 - u_1 \sin\alpha \operatorname{tg}\beta - u_2 \cos\alpha + u_3 \sin\alpha \sec\beta \\ A_3 p_3^{\bullet} &= (A_1 - A_2) p_1 p_2 + u_1 \cos\alpha \operatorname{tg}\beta - u_2 \sin\alpha - u_3 \cos\alpha \sec\beta \end{aligned} \tag{1.1}$$

$$\alpha_{i1}^{\bullet} = p_3 \alpha_{i2} - p_2 \alpha_{i3} \qquad (i = 1, 2, 3)\ (123)$$

$$\begin{aligned} A\alpha^{\bullet} &= G_1 + (G_2 \sin\alpha - G_3 \cos\alpha) \operatorname{tg}\beta - (A_1 + A) p_1 - \\ &\quad - (A_2 + A)(p_2 \sin\alpha - p_3 \cos\alpha) \operatorname{tg}\beta \\ A\beta^{\bullet} &= G_2 \cos\alpha + G_3 \sin\alpha - (A_2 + A) p_2 \cos\alpha - (A_3 + A) p_3 \sin\alpha \end{aligned}$$

$$G_i = \sum_k h_k \alpha_{ki}, \qquad h_i^{\bullet} = 0 \qquad (i, k = 1, 2, 3)$$

Here p_1, p_2, p_3 are the projections of the instantaneous angular velocity of the body on the axes $Ox_1x_2x_3$; A_1, A_2, A_3 are the body's moments of inertia relative to the mean axes $Ox_1x_2x_3$, tightly connected with it; A is the equatorial moment of inertia of the symmetric gyroscope; α is the rotation angle of the external ring relative to the body; β is that of the internal ring, counted from a plane perpendicular to the plane of the external ring, γ is the angle of rotor's natural rotation; u_i $(i = 1, 2, 3)$ are the guiding moments induced by the motors; α_{ik} $(i, k = 1, 2, 3)$ are the direction cosines between the axes of the mobile system of coordinates $Ox_1x_2x_3$ and the respective axes of the inertial system of coordinates $OX_1X_2X_3$; G_1, G_2, G_3, are the projections of the vector of system's kinetic moment on the axes $Ox_1x_2x_3$; h_1, h_2, h_3 are the constant projections of the vector of system's kinetic moment on the axes $OX_1X_2X_3$.

The equations of motion (1.1) are obtained in the assumption that $\cos\beta \neq 0$, i.e. the planes of the external and internal rings do not coincide, which is quite justified on the basis of the exposé presented below.

Eqs. (1.1) admit the particular solution

$$\begin{aligned} p_1 = \omega = \text{const}, \quad p_2 = p_3 = 0, \quad \alpha = -\omega t + \alpha_0, \quad \beta = \beta_0 \neq {}^1\!/_2\pi \\ \gamma^{\bullet} = \text{const}, \quad u_i = 0, \quad h_i = h_i^0 \qquad (i = 1, 2, 3) \end{aligned} \tag{1.2}$$

which represents the uniform rotation of the solid body around the main axis Ox_1 with angular velocity ω, while the external ring of the universal joint rotates with respect to the body with the same angular velocity in the opposite direction, whereupon the plane of the internal ring forms with that of the external ring a constant angle ${}^1\!/_2\pi - \beta_0$. Consequently, the stationary motion (1.2) means that the gyroscope effects relative to the body a regular precession, whereupon the rotor axis maintains an invariable direction in the inertial space.

Utilizing the expressions for G_i, brought out in the work [1], it is easy to see that the constants α_0, β_0 may be expressed by the initial values of the projections of system's h_i^0 kinetic moment, corresponding to the regime (1.2), in the form

$$\operatorname{tg} \alpha_0 = -\frac{h_2^\circ}{h_3^\circ}, \qquad \operatorname{tg} \beta_0 = \frac{h_1^\circ - A_1\omega}{\sqrt{h_2^{\circ 2} + h_3^{\circ 2}}} \qquad (h_2^{\circ 2} + h_3^{\circ 2} \neq 0)$$

When investigating the stability of solution (1.2) with respect to p_i, α_{ik} $(i, k = 1, 2, 3)$ mathematical difficulties might arise, inasmuch as some of the α_{ik} during the motion (1.2) clearly depend on time

$$\alpha_{11} = 1, \quad \alpha_{22} = \cos \omega t, \quad \alpha_{33} = \cos \omega t, \quad \alpha_{23} = -\sin \omega t$$
$$\alpha_{32} = \sin \omega t, \quad \alpha_{12} = \alpha_{21} = \alpha_{13} = \alpha_{31} = 0$$

and this is why the corresponding equations of perturbed motion, composed on the basis of the system of Eqs. (1.1), would contain periodical coefficients. In order to avoid the appearance of periodical coefficients, it is necessary to effect the passage from α_{ik} to new variables. With this in view we shall assume that the body is endowed with dynamic symmetry relative to the axis Ox_1, i.e. $A_2 = A_3$, and we shall introduce [2, 3] instead of the system of coordinates $Ox_1x_2x_3$ a new system of coordinates $Oz_1z_2z_3$, in which the axis Oz_1 coincides with the axis Ox_1, while axes Oz_2, Oz_3 lie in the equatorial plane of the body and do not participate in its rotary motion around the symmetry axis with the angular velocity ω. Let us denote by q_1, q_2, q_3 the projections of the instantaneous angular velocity of the system of coordinates $Oz_1z_2z_3$ on its axes

$$p_1 = q_1 + \omega, \quad p_2 = q_2 \cos \omega t + q_3 \sin \omega t, \quad p_3 = -q_2 \sin \omega t + q_3 \cos \omega t$$

We shall denote by G_1', G_2', G_3' the projections of the vector of system's kinetic moment on the axes $Oz_1z_2z_3$. Then, introducing the angles $\alpha_1 = \alpha + \omega t$, $\beta_1 = \beta$ and also the direction cosines β_{ik} $(i, k = 1, 2, 3)$ between the axes of coordinates $OX_1X_2X_3$ and $Oz_1z_2z_3$, we shall obtain in place of Eqs (1.1), the following ones:

$$\begin{aligned}
A_1 q_1^\cdot &= -u_1 \\
A_2 q_2^\cdot &= (A_3 - A_1) q_3 q_1 - A_1 \omega q_3 - u_1 \sin \alpha_1 \operatorname{tg} \beta_1 - u_2 \cos \alpha_1 + u_3 \sin \alpha_1 \sec \beta_1 \\
A_3 q_3^\cdot &= (A_1 - A_2) q_1 q_2 + A_1 \omega q_2 + u_1 \cos \alpha_1 \operatorname{tg} \beta_1 - u_2 \sin \alpha_1 - u_3 \cos \alpha_1 \sec \beta_1 \\
\beta_{i1}^\cdot &= q_3 \beta_{i2} - q_2 \beta_{i3} \qquad (i = 1, 2, 3) \\
A \alpha_1^\cdot &= G_1' + (G_2' \sin \alpha_1 - G_3' \cos \alpha_1) \operatorname{tg} \beta_1 - (A_1 + A) q_1 - \\
&\quad - A_1 \omega - (A_2 + A)(q_2 \sin \alpha_1 - q_3 \cos \alpha_1) \operatorname{tg} \beta_1 \\
A \beta_1^\cdot &= G_2' \cos \alpha_1 + G_3' \sin \alpha_1 - (A_2 - A)(q_2 \cos \alpha_1 + q_3 \sin \alpha_1)
\end{aligned} \qquad (1.3)$$

In the last relations (1.3)

$$G_i' = \sum_k h_k \beta_{ki}, \qquad h_k^{\cdot} = 0 \quad (i, k = 1, 2, 3)$$

Assuming as previously in [1] that $\beta_0 = 0$, i. e. $h_1^\circ = A_1\omega$, we shall rewrite the particular solution (1.2) in new denotations

$$q_i = 0, \quad \beta_{ik} = 1 \quad (i = k), \quad \beta_{ik} = 0 \quad (i \neq k), \quad \alpha_1 = \alpha_0 = \text{const}$$
$$\beta_1 = 0, \quad G_1' = A_1\omega, \quad G_j' = h_j^\circ = \text{const} \quad (j = 2, 3), \quad u_i = 0 \quad (i=1,2,3) \tag{1.4}$$

Let us assume the motion (1.4) as an unperturbed one and compose on the basis of (1.3) the equations of perturbed motion maintaining for perturbations the denotations of the initial variables. It is not difficult to verify that these equations have the following form:

$$q_1^{\cdot} = w_1, \quad q_2^{\cdot} = -\omega' q_3 + w_2 + Q_2, \quad q_3^{\cdot} = \omega' q_2 + w_3 + Q_3$$
$$\beta_{ii}^{\cdot} = B_{ii} \ (i = 1, 2, 3), \ \beta_{12}^{\cdot} = -q_3 + B_{12}, \quad \beta_{13}^{\cdot} = q_2 + B_{13} \ (123)$$
$$A\alpha^{\cdot} = h_1 + \sum_k (h_k^\circ + h_k)\beta_{k1} + \left\{\left[h_2^\circ + h_2 + \sum_k (h_k^\circ + h_k)\beta_{k2}\right] \times \right. \tag{1.5}$$
$$\left. \times \sin(\alpha_0 + \alpha_1) - \left[h_3^\circ + h_3 + \sum_k (h_k^\circ + h_k)\beta_{k3}\right]\cos(\alpha_0 + \alpha_1)\right\} \operatorname{tg}\beta_1 -$$
$$-(A_1 + A)q_1 - (A_2 + A)[q_2 \sin(\alpha_0 + \alpha_1) - q_3 \cos(\alpha_0 + \alpha_1)] \operatorname{tg}\beta_1$$
$$A\beta_1^{\cdot} = \left[h_2^\circ + h_2 + \sum_k (h_k^\circ + h_k)\beta_{k2}\right]\cos(\alpha_0 + \alpha_1) + \left[h_3^\circ + h_3 + \right. \tag{1.6}$$
$$\left. + \sum_k (h_k^\circ + h_k)\beta_{k3}\right]\sin(\alpha_0 + \alpha_1) - (A_2 + A)[q_2 \cos(\alpha_0 + \alpha_1) + q_3 \sin(\alpha_0 + \alpha_1)]$$
$$h_k^{\cdot} = 0 \quad (k = 1, 2, 3)$$
$$\omega' = (A_1 / A_2)\omega, \quad B_{i1} = q_3\beta_{i2} - q_2\beta_{i3} \quad (i = 1, 2, 3) \ (123)$$

Here Q_2, Q_3 denote the terms of an order of smallness not below the 2nd relative to q_i, u_i, α_1, β_1; h_k ($k = 1, 2, 3$) denote the initial perturbations of the system's kinetic moment, w_i are new guiding moments, linked with u_i by the relations

$$A_1 w_1 = -u_1, \quad A_2 w_2 = -u_2 \cos\alpha_0 + u_3 \sin\alpha_0$$
$$A_3 w_3 = -u_2 \sin\alpha_0 - u_3 \cos\alpha_0$$

Let us set up the following problem [1]: to determine the controls w_i in the form of functions of body's phase coordinates q_i, β_{ik}, such that the zero solution of the system of Eqs.(1.5), (1.6)

$$q_i = 0, \quad \beta_{ik} = 0, \quad h_k = 0 \quad (i, k = 1, 2, 3) \quad \alpha_1 = \beta_1 = 0 \tag{1.7}$$

be asymptotically stable with respect to q_i, $\dot{\beta}_{ik}$ and that at the same time the condition of minimum of the functional

$$\int_0^\infty \Omega(q_1, q_2, q_3, \beta_{11}, \beta_{12}, \ldots, \beta_{33}, w_1, w_2, w_3, \alpha_1, \beta_1)\, dt \tag{1.8}$$

be fulfilled. Here Ω is a certain specificially positive function with respect to q_i, β_{kh}, which shall be found in the process of problem's solution.

For the solution of the stated problem we shall take advantage of the basic theorem of the second Lyapunov method of investigation of problems of optimum stabilization [4], demonstrated by N. N. Krasovskiy. Inasmuch as the asymptotic stability of motion (1.7) may only be achieved by part of variables q_i, β_{ik} (phase coordinates of the body), one must also bear in mind the corresponding theorems established by V. V. Rumyantsev [5]. The impossibility of assuring with the aid of inner controls w_i the asymptotic stability of motion (1.7) by all variables of the system q_i, β_{ik}, h_k, α_1, β_1 is obvious from mechanical considerations: no inner forces can reduce to zero the perturbations of system's kinetic moment (body + gyroscope) h_h; however, by a corresponding selection of inner forces one may achieve the constancy of the kinetic moment of one part of the considered mechanical system (body) at the expense of the variation of the kinetic moment of its other part (gyroscope), and to assure, by the same token, the asymptotic stability of the system (1.7) by part of variables q_i, β_{ik}.

2. Analysis of Approximate Equations

The problem is resolved according to a scheme proposed in the work [1]. Let us consider at the outset the approximate system of equations obtained from Eqs.(1.5) at $Q_2 = Q_3 = 0$

$$\begin{gathered} q_1^{\cdot} = w_1, \quad q_2^{\cdot} = -\omega' q_3 + w_2, \quad q_3^{\cdot} = \omega' q_2 + w_3 \\ \beta_{ii}^{\cdot} = B_{ii} \quad (i = 1, 2, 3), \quad \beta_{12}^{\cdot} = -q_3 + B_{12}, \quad \beta_{13}^{\cdot} = q_2 + B_{13\,(1\,2\,3)} \end{gathered} \tag{2.1}$$

and study the question of optimum stabilization (in the above indicated sense) of the zero solution of Eqs.(2.1)

$$q_i = 0, \quad \beta_{ik} = 0 \; (i,\ k = 1,2,3) \tag{2.2}$$

The integrand Ω_1 is searched for in the form

$$\begin{aligned} \Omega_1 = F_1(q_1, q_2, q_3) + F_2(\beta_{11}, \beta_{12}, \ldots, \beta_{33}) + \sum n_i w_i^2 + \\ + \Lambda_1(q_1, q_2, q_3, \beta_{11}, \beta_{12}, \ldots, \beta_{33}, w_1, w_2, w_3) \end{aligned} \tag{2.3}$$

Here the function

$$F_1(q_1, q_2, q_3) = \sum_{i,h} e_{ik} q_i q_k$$

is a definitely positive form of q_i, the function F_2 is a definitely positive form of β_{ik}; the terms of higher order of smallness are denoted by Λ_1, the constant coefficients $n_i > 0$.

We construct the optimum Lyapunov function $V°$ in the form of linear combination

$$2V° = 2\Phi_0 + \sum_{i,k} k_i \Phi_{ii}. \qquad (2.4)$$

Here

$$2\Phi_0 = -2\sum_i k_i\beta_{ii} + \sum_i m_i q_i^2 + 2q_1\sum_{i,k} a_{ik}\beta_{ik} + 2q_2\sum_{i,k} b_{ik}\beta_{ik} + $$
$$+ 2q_3\sum_{i,k} c_{ik}\beta_{ik} \qquad (k_i > 0,\ m_i > 0)$$
$$\Phi_{hl} = \beta_{hl} + \beta_{lh} + \sum \beta_{hi}\beta_{li} \qquad (h, l = 1, 2, 3;\ h \leq l)$$

The last expressions represent the trivial integrals of Eqs.(1.5), (2.1). Applying the basic theorem [4], we obtain a system of algebraic equations linking the coefficients of functions Ω_1 and $V°$. These equations allow us to express all indeterminate coefficients a_{ik}, b_{ik}, c_{ik}, e_{ik} ($i, k = 1,2,3$) by the basic parameters k_i, m_i, n_i ($n = 1,2,3$)

Denoting

$$d_i = m_i/2n_i \ (i=1,2,3), \qquad 1/(d_2 d_3 + \omega'^2) = \mu,$$

we have

$$\begin{array}{lll} & a_{11} = b_{11} = c_{11} = 0 & \\ a_{12} = 0, & b_{12} = -\mu k_1\omega', & c_{12} = -\mu k_1 d_2 \\ a_{13} = 0, & b_{13} = \mu k_1 d_3, & c_{13} = -\mu k_1\omega' \\ a_{21} = 0, & b_{21} = \mu k_2\omega', & c_{21} = \mu k_2 d_2 \\ a_{22} = b_{22} = c_{22} = 0, & a_{23} = -k_2/d_1, & b_{23} = c_{23} = 0 \\ a_{31} = 0, & b_{31} = -\mu k_3 d_3, & c_{31} = \mu k_3\omega' \\ a_{32} = k_3/d_1, & b_{32} = c_{32} = 0, & a_{33} = b_{33} = c_{33} = 0 \end{array} \qquad (2.5)$$

The coefficients e_{ik} are found in the form

$$e_{11} = d_1^2 n_1 + a_{23} - a_{32}, \quad e_{22} = d_2^2 n_2 + b_{31} - b_{13}, \quad e_{33} = d_3^2 n_3 + c_{12} - c_{21}$$
$$2e_{12} = a_{21} - a_{13} - b_{32} + b_{23}, \quad 2e_{13} = a_{12} - a_{31} - c_{32} + c_{23} \qquad (2.6)$$
$$2e_{23} = \omega'(m_2 - m_3) + b_{12} - b_{21} - c_{13} + c_{31}$$

For the function $F_2(\beta_{11}, \beta_{12}, \ldots, \beta_{33})$ we have

$$F_2(\beta_{11}, \beta_{12}, \ldots, \beta_{33}) = \frac{1}{4n_1}\Big(\sum_{i,k} a_{ik}\beta_{ik}\Big)^2 + \frac{1}{4n_2}\Big(\sum_{i,k} b_{ik}\beta_{ik}\Big)^2 + \frac{1}{4n_3}\Big(\sum_{i,k} c_{ik}\beta_{ik}\Big)^2$$

Considering for simplicity

$$k_i = k, \quad m_i = m, \quad n_i = n, \quad d_i = d \quad (i = 1, 2, 3)$$

we obtain the controls

$$w_i = -\frac{1}{2n_i}\frac{\partial V^\circ}{\partial q_i} \quad (i = 1, 2, 3)$$

in the following form

$$\begin{aligned} w_1 &= -dq_1 + l_1(\beta_{23} - \beta_{32}) \\ w_2 &= -dq_2 + l_2(\beta_{31} - \beta_{13}) + l_3(\beta_{12} - \beta_{21}) \\ w_3 &= -dq_3 + l_2(\beta_{12} - \beta_{21}) - l_3(\beta_{31} - \beta_{13}) \end{aligned} \tag{2.7}$$

Here

$$l_1 = k/m, \quad l_2 = kd\mu/2n, \quad l_3 = k\omega'\mu/2n$$

Passing to initial control moments u_i, we find

$$\begin{aligned} u_1 &= A_1dq_1 + A_1l_1(\beta_{32} - \beta_{23}) \\ u_2 &= d(A_2q_2\cos\alpha_0 + A_3q_3\sin\alpha_0) + (\beta_{13} - \beta_{31})(A_2l_2\cos\alpha_0 - \\ &\quad - A_3l_3\sin\alpha_0) + (\beta_{21} - \beta_{12})(A_2l_2\sin\alpha_0 + A_3l_3\cos\alpha_0) \\ u_3 &= -d(A_2q_2\sin\alpha_0 - A_3l_3\cos\alpha_0) - (\beta_{13} - \beta_{31})(A_2l_2\sin\alpha_0 + \\ &\quad + A_3l_3\cos\alpha_0) + (\beta_{21} - \beta_{12})(A_2l_2\cos\alpha_0 - A_3l_3\sin\alpha_0) \end{aligned} \tag{2.8}$$

The integrand of the minimized integral will take the form

$$\begin{aligned} \Omega_1 = e_{11}q_1^2 + e_{22}(q_2^2 + q_3^2) + nl_1^2[(\beta_{32} - \beta_{23})^2 + d^2\mu(\beta_{13} - \beta_{31})^2 + \\ + d^2\mu(\beta_{21} - \beta_{12})^2] + \sum_i N_iu_i^2 + \Lambda_1 \end{aligned} \tag{2.9}$$

Here

$$e_{11} = (d^2 - 4l_1)n, \quad e_{22} = e_{33} = d^2n - 2kd\mu, \quad N_i = n/A_i^2 \quad (i = 1, 2, 3)$$
$$\Lambda_1 = -\sum_{i,k}(q_1a_{ik} + q_2b_{ik} + q_3c_{ik})B_{ik}$$

The main part of Ω_1 will be a definitely positive quadratic form from q_i, β_{ik}. The parameters k, m, n may, in particular, be chosen in such a way that the relations

$$e_{11} = A_1, \qquad c_{22} = A_2 = A_3.$$

be fulfilled.

In the first case the first three terms of (2.9) will constitute a doubled kinetic energy of the body

$$2T = \sum_i A_i q_i^2.$$

Let us now pass from dependent variables β_{ik} to two independent Krylov angles θ, ψ, determining the position of the nonrotating system of coordinates $Oz_1z_2z_3$ relative to the inertial system of coordinates $OX_1X_2X_3$, having taken OX_3 and Ox_1 for the basic axes and postulating in the table of direction cosines $\phi = 0$. Then, for u_i we shall obtain (refer to [6])

$$\begin{aligned} u_1 &= A_1 d q_1 + U_1(\theta, \psi) \\ u_2 &= d(A_2 q_2 \cos\alpha_0 + A_3 q_3 \sin\alpha_0) + 2\theta(A_2 l_2 \cos\alpha_0 - \\ &\quad - A_3 l_3 \sin\alpha_0) + 2\psi(A_2 l_2 \sin\alpha_0 + A_3 l_3 \cos\alpha_0) + U_2(\theta, \psi) \\ u_3 &= -d(A_2 q_2 \sin\alpha_0 - A_3 l_3 \cos\alpha_0) - 2\theta(A_2 l_2 \sin\alpha_0 + \\ &\quad + A_3 l_3 \cos\alpha_0) + 2\psi(A_2 l_2 \cos\sigma_0 - A_3 l_3 \sin\alpha_0) + U_3(\theta, \psi) \end{aligned} \tag{2.10}$$

Here the functions $U_i(\theta, \psi)$, $(i = 1,2,3)$ begin from the terms of second order of smallness relative to θ, ψ. For the function Ω_1 we have

$$\Omega_1 = e_{11} q_1^2 + e_{22}(q_2^2 + q_3^2) + a(\theta^2 + \psi^2) + \sum_i N_i u_i^2 + \ldots \tag{2.11}$$

The multiple-point imply the terms of higher order of smallness with respect to θ, ψ, and the following denotation:

$$a = 4 n l_1^2 d^2 \mu$$

is introduced.

3. Analysis of Total Equations

We shall now establish that the optimum Lyapunov function (2.4) and the optimum control (2.7), (2.8), found for the approximate system of Eqs.(2.1), assure the optimum stabilization of motion (1.7) conforming to q_i, β_{ik} on the strength of total equations (1.5), (1.6). Upon substitution of expressions (2.8) into Eqs.(1.5), the functions Q_2, Q_3 will assume the form

$$\begin{aligned} A_2 Q_2 &= (A_3 - A_1) q_3 q_1 + Q_2^*(u_1, u_2, u_3, \alpha_1, \beta_1) \\ A_3 Q_3 &= (A_1 - A_2) q_1 q_2 + Q_3^*(u_1, u_2, u_3, \alpha_1, \beta_1) \end{aligned}$$

Here Q_2^*, Q_3^* are linear functions of u_1, u_2, u_3 (and consequently of q_i, β_{ik}) with coefficients being analytical functions of α_1, β_1 and becoming zero at $\alpha_1 = \beta_1 = 0$. The total derivative in time from function (2.4), will have, on the strength of Eqs. (1.5), (1.6), the form

$$\Omega = \Omega_1 + \Omega_2 + \Omega_3 \tag{3.1}$$

The function Ω_1 is determined by the expression (2.9), Ω_2 represents an alternating quadratic form from q_i, β_{ik} with coefficients depending on α_1, β_1.

$$\Omega_2 = \sum_{i,k} g_{ik}(\alpha_1, \beta_1) q_i \beta_{ik} \tag{3.2}$$

while the terms of third order of smallness conforming to q_i, β_{ik}, not influencing the sign of Ω, are denoted by Ω_3. According to a well known theorem (ref. to page 31 of [4]), the sum of quadratic forms of Ω_1 and Ω_2 (and together with it the whole function Ω) is definitely positive conforming to q_i, β_{ik}, provided the coefficients $g_{ik}(\alpha_1, \beta_1)$ are sufficiently small. Obviously, the latter takes place provided the motion (1.7) is steady conforming to α_1, β_1.

The demonstration of motion (1.7) stability conforming to α_1, β_1 is conducted on the basis of the theorem of [7], according to which the problem is reduced to the study of stability of the zero solution

$$\alpha_1 = \beta_1 = 0, \qquad h_k = 0 \qquad (k=1,2,3) \tag{3.3}$$

of the "shortened" system of equations

$$\begin{aligned} A\alpha_1^{\cdot} &= h_1 + [(h_2^\circ + h_2)\sin(\alpha_0 + \alpha_1) - (h_3^\circ + h_3)\cos(\alpha_0 + \alpha_1)]\,\mathrm{tg}\,\beta_1 \\ A\beta_1^{\cdot} &= (h_2^\circ + h_2)\cos(\alpha_0 + \alpha_1) + (h_3^\circ + h_3)\sin(\alpha_0 + \alpha_1) \\ h_k^{\cdot} &= 0 \qquad (k=1,2,3) \end{aligned} \tag{3.4}$$

obtained from (1.6) at

$$q_i = 0, \quad \beta_{ik} = 0 \qquad (i, k = 1, 2, 3)$$

The stability of the solution (3.3) of the system (3.4) was established earlier [1]. Consequently, the control (2.7), (2.8), really resolves the problem of optimum stabilization of motion (1.7) relative to q_i, β_{ih} on the strength of total equations (1.5), (1.6), whereupon the integral of the minimized functional (1.8) is determined in the form (3.1).

The fact of motion's (1.7) stability relative to α_1, β_1 demonstrates the validity of the assumption $\cos\beta \neq 0$, made during the deriving of Eqs. (1.1).

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Comparing the optimum control (2.8) with the corresponding control obtained in the work [3], let us note that control (2.8) is simpler, and consequently, more practical from the standpoint of technical realization. This simplicity is due to the fact that control (2.8) is not explicitly dependent on time and on the initial perturbations of the kinetic moment of system (1.1). The latter is the consequence of the fact that the initial equations of body motion (the first three equations of system (1.1) or (1.3)) do not contain constants h_k. The noted case did not take place in [3].

A practical realization of the found control (2.8), (2.10) is conceived as follows. So long as the body effects a rotary motion (1.4) and the external perturbations are absent, the controlling (guiding) motors are switched off ($u_i = 0$). As soon as initial perturbations h_i appear, the body emerges from the stationary regime (1.4), acquiring displacement by angular velocity (q_1, q_2, q_3) and angles (θ, ψ). Special devices must measure these perturbations of body's phase coordinates q_1, q_2, q_3, θ, ψ and feed the corresponding signals to control motors. The latter act upon the gyroscope, forming on the basis of obtained signals the moments u_1, u_2, u_3 in the form (2.10).

**** T H E E N D ****

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